Modal Logics with Transitive Closure: Completeness, Decidability, Filtration

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Abstract

We give a sufficient condition for Kripke completeness of modal logics that have the transitive closure modality. More precisely, we show that if a modal logic admits what we call *definable filtration*, then its enrichment with the transitive closure modality (and the corresponding axioms) is Kripke complete; in addition, the resulting logic has the finite model property and admits definable filtration, too. This argument can be iterated, and as an application we obtain the finite model property for PDL-like expansions of multimodal logics that admit definable filtration.

Keywords: Filtration, decidability, finite model property, transitive closure, PDL.

Introduction

This paper makes a contribution to the study of modal logics enriched by the transitive closure modality.

Modal logics that, in addition to the modal operator $\Box$ for a binary relation R, also contain the operator $\boxplus$ for the transitive closure of R, are quite common [8]. For instance, such are the propositional dynamic logic (PDL) [7] or von Wright's logic $\mathrm{Log}(\mathbb{N}, \mathsf{succ}, <)$ (see [18]). Other examples include logics

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with the operator of common knowledge and 'everyone knows that' in epistemic logics [6] such as the logic Team for collective beliefs and actions [5].

So far, completeness and decidability results for such logics have had bespoke proofs, though many of them are based on Segerberg's [16] and Kozen and Parikh's arguments for PDL [13]. In this paper we present a toolkit for obtaining results on completeness, finite model property, and decidability applicable to a wide range of modal logics with transitive closure.

In Section 2, we recall a (rather general) notion of filtration and come up with a hierarchy of 'admits filtration' notions (including those studied in our earlier work [12]). Section 3 contains our main result (announced in [20]): if the class of models of a logic L *admits definable filtration* (see Definition 2.5), then the axioms of L together with Segerberg's axioms for the transitive closure modality yield a complete axiomatization of the bimodal logic of the class of frames for L augmented by the transitive closure of the accessibility relation. Moreover, the resulting logic has the finite model property (and is decidable, if L was finitely axiomatizable). Section 4 presents examples of logics that satisfy our sufficient condition; we also show how this condition can be 'iterated' for obtaining completeness for 'PDLizations' of a family of logics.

1 Preliminaries

We assume the reader to be familiar with syntax and semantics of multi-modal logic [1,3], so we only briefly recall some notions and fix notation. Let Σ be a (usually finite) alphabet (of indices for modalities). The set $\mathsf{Fm}(\Sigma)$ of *modal formulas* (*over* Σ) is defined from propositional letters $\mathsf{Var} = \{p_0, p_1, \ldots\}$ using Boolean connectives and the modalities $[e]$, for $e \in \Sigma$, according to the syntax:

$$\varphi \ ::= \ \bot \ \mid \ p_i \ \mid \ \varphi \to \psi \ \mid \ [e]\,\varphi.$$

We use standard abbreviations (e.g., $\top, \wedge$); in particular, $\langle e\rangle\varphi := \neg\,[e]\,\neg\varphi$. For a set of formulas Γ, by $\mathsf{Sub}(\Gamma)$ we denote the set of all subformulas of formulas from Γ. We say that Γ is Sub-*closed* if $\mathsf{Sub}(\Gamma) \subseteq \Gamma$.

A (Σ-)*frame* is a pair $F = (W, (R_e)_{e\in\Sigma})$, where $W \neq \varnothing$ and $R_e \subseteq W \times W$ for $e \in \Sigma$. A *model* based on F is a pair $M = (F, V)$, where $V(p) \subseteq W$, for all $p \in \mathsf{Var}$. The *truth relation* $M, x \models \varphi$ is defined in the usual way, e.g.

$$M, x \models [e]\,\varphi \quad \leftrightharpoons \quad \text{for all } y \in W, \ \text{ if } x\, R_e\, y \ \text{ then } \ M, y \models \varphi.$$

We write $M \models \varphi$ if $M, x \models \varphi$ for all x in M. A formula φ is *valid* on F, notation $F \models \varphi$, if $M \models \varphi$ for all M based on F. For a class of frames $\mathcal{F}$, an $\mathcal{F}$-*model* is a model based on a frame from $\mathcal{F}$.

A (*normal modal*) *logic* (*over* Σ) is a set of formulas L that contains all classical tautologies, the axioms $[e](p \to q) \to ([e]\,p \to [e]\,q)$, for each $e \in \Sigma$, and is closed under the rules of modus ponens, substitution, and necessitation (from φ, infer $[e]\,\varphi$, for each $e \in \Sigma$). An L-*frame* is a frame F such that $F \models L$. *The logic of a class of frames* $\mathcal{F}$ is the set of all formulas that are valid in $\mathcal{F}$. A logic L is (*Kripke*) *complete* if it is the logic of some class of frames. A logic L has the *finite model property* (FMP) if it is the logic of some class of finite frames; or equivalently (see, e.g., [1, Th. 3.28]) if, for every formula $\varphi \notin L$, there is a finite model M such that $M \models L$ and $M \not\models \varphi$. For a logic L, put

$$\mathsf{Fr}(L) = \{ F \mid F \text{ is a frame and } F \models L\},$$
$$\mathsf{Mod}(L) = \{ M \mid M \text{ is a model and } M \models L\}.$$

Clearly, every $\mathsf{Fr}(L)$-model belongs to $\mathsf{Mod}(L)$. The converse does not hold in general; e.g., the canonical model of a non-canonical logic L is not a $\mathsf{Fr}(L)$-model. But the converse holds in the following special case. A model M is called *differentiated* if any two points in M can be distinguished by a formula.

Lemma 1.1 (See e.g. [10, Ex. 4.9]) *Let $M = (F, V)$ be a finite differentiated model. If all substitution instances of a formula φ are true in M, then $F \models \varphi$.*

In particular, if $M \models L$, where L is a logic, then $F \models L$.

Harrop's theorem. *A finitely axiomatizable logic with the FMP is decidable.*

2 Filtration

The notion of a filtration we introduce below slightly generalizes the standard one (cf. [1, Def. 2.36], [3, Sect. 5.3]) in the following aspect: given a finite set of formulas Γ, we define a filtration as a model obtained by factoring a given model through an equivalence relation that we allow to be *finer* than the one induced by Γ. This modification seems to first appear in [21]; see also [22,23].

Let $M = (W, (R_e)_{e\in\Sigma}, V)$ be a model and Γ a finite Sub-closed set of Σ-formulas. An equivalence relation $\sim$ on W is *of finite index* if the quotient set $W/{\sim}$ is finite. The equivalence relation *induced by* Γ is defined as follows:

$$x \sim_\Gamma y \quad \leftrightharpoons \quad \forall \varphi \in \Gamma \ \big(M, x \models \varphi \ \Leftrightarrow \ M, y \models \varphi\big).$$

Clearly, $\sim_\Gamma$ is of finite index. We say that an equivalence relation $\sim$ *respects* Γ if $\sim \subseteq \sim_\Gamma$; in other words, if for every class $[x]_\sim \subseteq W$ and every formula $\varphi \in \Gamma$, φ is either true in all points of $[x]_\sim$ or false in all points of $[x]_\sim$.

Definition 2.1 (Filtration) By a *filtration* of a model M that *respects* a set of formulas Γ (or a Γ-*filtration of* M) we mean any model $\widehat{M} = (\widehat{W}, (\widehat{R}_e)_{e\in\Sigma}, \widehat{V})$ that satisfies the following conditions:

- $\widehat{W} = W/{\sim}$, for some equivalence relation of finite index $\sim$ on W;
- the equivalence relation $\sim$ respects Γ, i.e., $x \sim y$ implies $x \sim_\Gamma y$;
- the valuation $\widehat{V}$ is defined on the variables $p \in \Gamma$ canonically: $\widehat{x} \models p \Leftrightarrow x \models p$, for all points $x \in W$, where $\widehat{x} := [x]_\sim$ denotes the $\sim$-class of a point x;
- $R^{\mathsf{min}}_{\sim,e} \subseteq \widehat{R}_e \subseteq R^{\mathsf{max}}_{\Gamma,e}$, for each $e \in \Sigma$. Here $R^{\mathsf{min}}_{\sim,e}$ is the e-th *minimal filtered relation* on $\widehat{W}$, and $R^{\mathsf{max}}_{\Gamma,e}$ is the e-th *maximal filtered relation* on $\widehat{W}$ induced by the set of formulas Γ; they are defined in the usual way:

$$\widehat{x}\, R^{\mathsf{min}}_{\sim,e}\, \widehat{y} \quad \leftrightharpoons \quad \exists x' \sim x \ \exists y' \sim y\colon \ x'\, R_e\, y',$$
$$\widehat{x}\, R^{\mathsf{max}}_{\Gamma,e}\, \widehat{y} \quad \leftrightharpoons \quad \text{for every formula } [e]\varphi \in \Gamma \ \big(M, x \models [e]\varphi \ \Rightarrow \ M, y \models \varphi\big).$$

If $\sim = \sim_\Phi$ for a finite set of formulas Φ, then we call $\widehat{M}$ a *definable* Γ-*filtration* of M (*through* Φ); we can assume, without loss of generality, that $\Phi \supseteq \Gamma$.

Note that the relations $R^{\mathsf{min}}_{\sim,e}$ and $R^{\mathsf{max}}_{\Gamma,e}$ are well-defined and $R^{\mathsf{min}}_{\sim,e} \subseteq R^{\mathsf{max}}_{\Gamma,e}$. The condition $R^{\mathsf{min}}_{\sim,e} \subseteq \widehat{R}_e$ can be rewritten as $\forall x, y \in W\ \ (x\, R_e\, y \Rightarrow \widehat{x}\, \widehat{R}_e\, \widehat{y})$. A filtration is always a finite model. The following is the key lemma about filtration (cf. [1, Th. 2.39], [3, Th. 5.23]).

Lemma 2.2 (Filtration lemma) *Suppose that Γ is a finite* Sub*-closed set of formulas and $\widehat{M}$ is a Γ-filtration of a model M. Then, for all points $x \in W$ and all formulas $\varphi \in \Gamma$, we have: $M, x \models \varphi \;\Leftrightarrow\; \widehat{M}, \widehat{x} \models \varphi$.*

2.1 Admissibility of filtration

Definition 2.3 (ADF for classes of frames) We say that a class of frames $\mathcal{F}$ *admits (definable) filtration* if, for any finite Sub-closed set of formulas Γ and an $\mathcal{F}$-model M, there exists an $\mathcal{F}$-model that is a (definable) Γ-filtration of M.

It is well-known that filtration (of the class of all frames) is a method of proving the FMP for complete modal logics; let us state this explicitly.

Lemma 2.4 (AF for frames implies FMP) *If the class of its frames* $\mathsf{Fr}(L)$ *admits filtration and the logic L is Kripke complete, then L has the FMP.*

Proof. If $\varphi \notin L$, then, by completeness of L, there is a frame $F \models L$ with $F \not\models \varphi$. Taking $\Gamma = \mathsf{Sub}(\varphi)$ and Γ-filtrating the model based on F that falsifies the formula φ, we obtain a finite frame $F' \models L$ with $F' \not\models \varphi$. □

Definition 2.5 (ADF for classes of models) We say that a class of models $\mathcal{M}$ *admits (definable) filtration* if, for any finite Sub-closed set of formulas Γ and any $M \in \mathcal{M}$, there is a model in $\mathcal{M}$ that is a (definable) Γ-filtration of M.

The next lemma shows that filtration of the class of all models $\mathsf{Mod}(L)$ is a method of obtaining Kripke (frame!) completeness (and FMP, of course).

Lemma 2.6 (AF for models implies FMP) *If the class of models* $\mathsf{Mod}(L)$ *of a logic L admits filtration, then L has the FMP and hence is Kripke complete.*

Proof. Any normal logic L is model-complete, i.e., $\varphi \in L$ iff $\mathsf{Mod}(L) \models \varphi$; moreover, L is complete w.r.t. a single, *canonical* model M_L. Therefore, if $\varphi \notin L$, then there is a model $M \in \mathsf{Mod}(L)$ such that $M \not\models \varphi$. Let $\Gamma = \mathsf{Sub}(\varphi)$. Take a Γ-filtration $\widehat{M}$ of M such that $\widehat{M} \models L$; here $\widehat{M}$ is a finite model. By taking the filtration of $\widehat{M}$ through the set of all formulas Fm, we obtain a finite differentiated model $\widehat{M}' = (\widehat{F}', \widehat{V}')$ modally equivalent to $\widehat{M}$. So $\widehat{M}' \models L$ and $\widehat{M}' \not\models \varphi$. By Lemma 1.1, $\widehat{F}' \models L$. Thus, every non-theorem of L is falsified in some finite L-frame. So, L is Kripke (frame) complete and even has the FMP. □

So far, we are not aware of any example of a logic whose class of frames (or models) admits filtration, but not definable filtration.

We have two variants of the notion "a logic L admits (definable) filtration":

(I) the class of frames $\mathsf{Fr}(L)$ admits (definable) filtration (Definition 2.3);

(II) the class of models $\mathsf{Mod}(L)$ admits (definable) filtration (Definition 2.5).

In both variants, we filtrate a model $M = (F, V)$ into a model $\widehat{M} = (\widehat{F}, \widehat{V})$. The precondition $(F \models L)$ in (I) is stronger than that $(M \models L)$ in (II). The postcondition ($\widehat{F} \models L$) in (I) is stronger than ($\widehat{M} \models L$) in (II), too. However, we can always make sure that the finite model $\widehat{M}$ is differentiated. Then $\widehat{M} \models L$ iff $\widehat{F} \models L$. Thus, (II) is stronger than (I). Let us state this explicitly.

Lemma 2.7 (ADF for models implies ADF for frames)
For any logic L, if $\mathsf{Mod}(L)$ admits (definable) filtration, then so does $\mathsf{Fr}(L)$.

Proof. Take any finite Sub-closed set of formulas Γ and a model $M = (F, V)$ with $F \models L$. Then $M \in \mathsf{Mod}(L)$. By assumption, the model M has a (definable) Γ-filtration $\widehat{M} = (\widehat{F}, \widehat{V})$ with $\widehat{M} \models L$. The model $\widehat{M}$ is finite and, without loss of generality, differentiated, by Lemma A.2 (in Appendix). Then $\widehat{F} \models L$, by Lemma 1.1. Thus, $\mathsf{Fr}(L)$ admits (definable) filtration. □

The converse implication in the above lemma does not hold in general. Consider the logic **Ver** of the irreflexive singleton frame; it is axiomatized by $\Box\bot$. One can easily see that the class of its frames $\mathcal{F}$ admits definable filtration. But there are continuum many other logics L with the same class of frames $\mathcal{F}$ (cf. [2], [3, Ex. 10.57]), so that the class *frames* $\mathsf{Fr}(L)$ admits definable filtration, too. However, each of these logics is Kripke incomplete and, by Lemma 2.6, the class of *models* $\mathsf{Mod}(L)$ does not admit filtration.

Next we prove that, for the *canonical* logics, the converse implication holds, and so the notions (I) and (II) coincide, if we consider *definable* filtration. To simplify notation, we work with the unimodal case. Recall that one can build the *canonical* frame $F_T = (W_T, R_T)$ and *canonical* model $M_T = (F_T, V_T)$ not only for a (consistent) normal *logic*, but more generally for a *normal theory* T (which contains all theorems of $\mathbf{K}$ and is closed under monus ponens and necessitation). Any point $x \in W_T$ is a consistent (never $A, \neg A \in x$) complete (always $A \in x$ or $\neg A \in x$) theory (i.e., closed under modus ponens) containing T.

A logic L is called *canonical* if $F_L \models L$. The following is a well-known fact.

Lemma 2.8 (Canonical generated submodel) *If $T \subseteq T'$ are consistent normal theories, then $M_{T'}$ is a generated submodel of M_T. Similarly for frames.*

Proof. Assume $x \in W_{T'}$, $y \in W_T$, and $x\, R_T y$. To prove that $y \in W_{T'}$, i.e., $T' \subseteq y$, take any formula $A \in T'$. By normality $\Box A \in T'$. Since $T' \subseteq x$, we have $\Box A \in x$. By definition of R_T, we obtain $A \in y$. □

A typical example of a normal theory is the theory of a model $T = \mathsf{Th}(M)$. For a model $M = (W, R, V)$, consider the canonical model M_T of its theory and the *canonical mapping* t from M to M_T defined, for $a \in W$, by

$$t(a) = \mathsf{Th}(M, a) \in W_T.$$

It is monotonic ($a\, R\, b \Rightarrow t(a)\, R_T\, t(b)$), but in general it is neither surjective, nor a p-morphism. The following lemma (proved in Appendix, see Lemma A.3) shows what happens to the canonical mapping if we filtrate both M and M_T through a finite set of formulas Φ.

Lemma 2.9 *Under the above conditions, any finite set of formulas Φ induces a bijection between the quotient sets $W/{\sim_\Phi}$ and $W_T/{\sim_\Phi}$ defined, for $a \in W$, by*

$$f([a]_{\sim_\Phi}) \;:=\; [t(a)]_{\sim_\Phi}.$$

Theorem 2.10 (ADF for frames implies ADF for models) *If L is a canonical logic, then* $\mathsf{Mod}(L)$ *admits definable filtration iff so does* $\mathsf{Fr}(L)$.

Proof. (⇒) By Lemma 2.7. (⇐) IDEA: in order to filtrate a model $M \models L$, we filtrate the canonical model M_T of its theory $T = \mathsf{Th}(M)$ and then use the bijection from Lemma 2.9 to transfer the filtration back to M.

Take a finite Sub-closed set of formulas Γ and a model $M = (W, R, V)$ with $M \models L$. Its theory $T = \mathsf{Th}(M)$ contains L, hence F_T is a generated subframe of F_L, by Lemma 2.8. Since L is canonical, we have $F_L \models L$ and so $F_T \models L$. Thus, M_T is a $\mathsf{Fr}(L)$-model and, by assumption, we can filtrate it.

Therefore, the model M_T has a Γ-filtration $\widehat{M_T} = (\widehat{W_T}, \widehat{R_T}, \widehat{V_T})$ (through some finite set of formulas $\Phi \supseteq \Gamma$) with $\widehat{F_T} \models L$. By Lemma 2.9, there is a bijection f between the finite sets $\widehat{W} = (W/{\sim_\Phi})$ and $\widehat{W_T} = (W_T/{\sim_\Phi})$. Now we build a model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{V})$ isomorphic to $\widehat{M_T}$, by putting, for all $a, b \in W$:

$$\widehat{a}\,\widehat{R}\,\widehat{b} \text{ iff } f(\widehat{a})\,\widehat{R_T}\,f(\widehat{b}); \qquad \widehat{a} \models p \text{ iff } f(\widehat{a}) \models p, \text{ for all variables } p \in \Gamma.$$

Since the frames $\widehat{F}$ and $\widehat{F_T}$ are isomorphic and $\widehat{F_T} \models L$, we have $\widehat{F} \models L$. It remains to prove that $\widehat{M}$ is a Γ-filtration (through Φ) of M. Below, we denote $x = t(a) = \mathsf{Th}(M, a)$ and $y = t(b) = \mathsf{Th}(M, b)$, so that $f(\widehat{a}) = \widehat{x}$ and $f(\widehat{b}) = \widehat{y}$.

(var) Let us check that $\widehat{M}, \widehat{a} \models p$ iff $M, a \models p$, for all $p \in \Gamma$. We have:

$$\widehat{M}, \widehat{a} \models p \;\Leftrightarrow\; \widehat{M_T}, \widehat{x} \models p \;\Leftrightarrow\; M_T, x \models p \;\Leftrightarrow\; p \in x \;\Leftrightarrow\; M, a \models p.$$

(min) Let us check that $R^{\mathsf{min}}_{\sim_\Phi} \subseteq \widehat{R}$, i.e., $\forall a, b \in W\ (a\,R\,b \Rightarrow \widehat{a}\,\widehat{R}\,\widehat{b})$.

We use the monotonicity of $t(\cdot)$ and the condition (min) for $\widehat{R_T}$:

$$a\,R\,b \;\Longrightarrow\; t(a)\,R_T\,t(b) \;\Leftrightarrow\; x\,R_T\,y \;\Longrightarrow\; \widehat{x}\,\widehat{R_T}\,\widehat{y} \;\Leftrightarrow\; \widehat{a}\,\widehat{R}\,\widehat{b}.$$

(max) Let us check that $\widehat{R} \subseteq R^{\mathsf{max}}_\Gamma$. Assume $\widehat{a}\,\widehat{R}\,\widehat{b}$. Then $\widehat{x}\,\widehat{R_T}\,\widehat{y}$.

By the condition (max) for $\widehat{R_T}$, we have $\widehat{x}\,((R_T)^{\mathsf{max}}_\Gamma)\,\widehat{y}$.

We need to show that $\widehat{a}\,R^{\mathsf{max}}_\Gamma\,\widehat{b}$. For any formula $\Box A \in \Gamma$, we have:

$$M, a \models \Box A \Leftrightarrow \Box A \in x \Leftrightarrow M_T, x \models \Box A \Rightarrow M_T, y \models A \Leftrightarrow A \in y \Leftrightarrow M, b \models A.$$

This completes the proof of the theorem. □

3 Logics with the transitive closure modality

In this section, $L \subseteq \mathsf{Fm}(\Box)$ is a normal unimodal logic. Let $L^{\boxplus} \subseteq \mathsf{Fm}(\Box, \boxplus)$ be the minimal normal logic that extends L with the following axioms describing the interaction between the modality $\Box$ and the *transitive closure* modality $\boxplus$:

$$\text{(A1) } \boxplus p \to \Box p, \qquad \text{(A2) } \boxplus p \to \Box \boxplus p, \qquad \text{(A3) } \boxplus(p \to \Box p) \to (\Box p \to \boxplus p).$$

Segerberg [16] (see also [17,19]) and later Kozen and Parikh [13] proved that the logic $\mathbf{K}^{\boxplus}$ (and even PDL) is complete and has the FMP; in other words, it is the logic of finite frames of the form (W, R, R^+); hence it is decidable (more exactly, EXPTIME-complete); see also [4] for a constructive variant of completeness theorem. The logic $\mathbf{K}^{\boxplus}$ is known to be not canonical (see Lemma A.5 in Appendix). Thus, even for simple logics we cannot use canonical models as a method of obtaining completeness.

To the best of our knowledge, up to now, there were no general results on the completeness and decidability for the $\boxplus$-companions of logics other than $\mathbf{K}$. Here we obtain one such result. We give a condition on L sufficient for the completeness of $L^{\boxplus}$. The condition is strong enough and guarantees not only the completeness, but the FMP of $L^{\boxplus}$; this limits the scope of our approach.

For simplicity, in this section we assume that L is unimodal. The results transfer easily to multi-modal logics. Given a unimodal frame $F = (W, R)$, we denote $F^{\oplus} = (W, R, R^+)$. Given a class of unimodal frames $\mathcal{F}$, we denote $\mathcal{F}^{\oplus} = \{F^{\oplus} \mid F \in \mathcal{F}\}$. Similarly for a model $M^{\oplus}$ and a class of models $\mathcal{M}^{\oplus}$.

Lemma 3.1 $(W, R, S) \models \{(\mathsf{A1}), (\mathsf{A2}), (\mathsf{A3})\}$ *iff* $R^+ = S$.

Proof. This is a known fact. Lemma A.4 (in Appendix) gives more details. □

Lemma 3.2 **(a)** $\mathsf{Mod}(L)^{\oplus} \subseteq \mathsf{Mod}(L^{\boxplus})$. **(b)** $\mathsf{Fr}(L)^{\oplus} = \mathsf{Fr}(L^{\boxplus})$.

Proof. Any frame of the form (W, R, R^+) validates (A1), (A2), and (A3). □

Lemma 3.3 (Conservativity) *For any consistent normal logic L, the logic $L^{\boxplus}$ is a conservative extension of L: if $A \in \mathsf{Fm}(\Box)$ and $L^{\boxplus} \vdash A$, then $L \vdash A$.*

Proof. If $L \nvdash A$, then $M_L \not\models A$ and $M_L^{\oplus} \not\models A$. But $M_L^{\oplus} \models L^{\boxplus}$. So $L^{\boxplus} \nvdash A$. □

3.1 Completeness for logics with the transitive closure modality

In the proof of the main result, we will need to modify a valuation *definably*. By φ^{σ} we denote the application of a substitution $\sigma \colon \mathsf{Var} \to \mathsf{Fm}$ to a formula φ.

Definition 3.4 By a (*modally*) *definable variant* of a model $M = (F, V)$ we mean a model of the form $M^{\sigma} = (F, V^{\sigma})$, for some substitution σ, where the valuation V^{σ} is defined by $V^{\sigma}(p) = V(p^{\sigma})$, for every variable p.

In other words, $M^{\sigma}, x \models p$ iff $M, x \models p^{\sigma}$. By induction one can easily prove:

Lemma 3.5 $M^{\sigma}, x \models \varphi$ *iff* $M, x \models \varphi^{\sigma}$, *for all formulas* φ.

Since a logic is closed under substitutions, we obtain the following fact.

Lemma 3.6 *If L is a logic and $M \models L$, then $M^{\sigma} \models L$, for any substitution σ.*

Recall that the formulas (A1) and (A2) are *canonical*, so they are valid on the canonical frame of $L^{\boxplus}$, for any L. For (A3), this is not the case even for the case $L = \mathbf{K}$: in the canonical frame $F_{\mathbf{K}^{\boxplus}} = (W, R, S)$, only a strict inclusion $R^+ \subsetneq S$ holds (see lemma A.5 in Appendix).

However, in order to obtain the completeness of $L^{\boxplus}$, we do not necessarily need the converse inclusion $S \subseteq R^+$ in the canonical frame of $L^{\boxplus}$. Instead, we do a walk around: given any model $M = (W, R, S, V)$ of $L^{\boxplus}$ (e.g., its canonical model), we remove S, filtrate (W, R, V) into a finite model $(\widehat{W}, \widehat{R}, \widehat{V})$, and then augment it with $(\widehat{R})^+$. It only remains to prove that the resulting finite bi-modal model is a definable filtration of the original bi-modal model M; i.e., that $(\widehat{R})^+$ is between the minimal and the maximal filtered relations. For maximal, the inclusion follows from the axioms (A1) and (A2) only (see (7) in the proof of Theorem 3.8 below); on the contrary, for the minimal, the required inclusion (see (5) in that proof) holds due to the following remarkable property of (A3).

Let us write $M \models A^*$ if we have $M \models A^\sigma$ for all substitutions σ.

Lemma 3.7 (Induction axiom and minimal filtration)
Let $M = (W, R, S, V) \models (\mathsf{A3})^*$ and let $\Phi \subseteq \mathsf{Fm}$ be finite. Then $S^{\mathsf{min}}_{\sim_\Phi} \subseteq (R^{\mathsf{min}}_{\sim_\Phi})^+$.

Proof. Denote $r := R^{\mathsf{min}}_{\sim_\Phi}$ and $s := S^{\mathsf{min}}_{\sim_\Phi}$. To prove $s \subseteq r^+$, assume $\widehat{x}\, s\, \widehat{y}$. By definition of the minimal filtered relation $S^{\mathsf{min}}_{\sim_\Phi}$, we can assume, without loss of generality, that $x\, S\, y$. Consider $Y := r^+(\widehat{x}) \subseteq \widehat{W}$. We need to show that $\widehat{y} \in Y$.

Since Φ is finite, every $\sim_\Phi$-equivalence class $\widehat{z} \subseteq W$ is a definable (by some formula) subset of W. Since Y is a *finite* collection of such subsets, their union $\bigcup Y \subseteq W$ is also a definable subset of W. So, there is a formula φ such that, for all $z \in W$, we have: $M, z \models \varphi$ iff $z \in \bigcup Y$ iff $\widehat{z} \in Y$.

Firstly, $M \models \varphi \to \Box\varphi$. Indeed, if $M, a \models \varphi$, $a\, R\, b$, then $\widehat{a} \in Y$, $\widehat{a}\, r\, \widehat{b}$. But Y is closed under r, hence $\widehat{b} \in Y$ and $M, b \models \varphi$. Therefore, $M \models \boxplus(\varphi \to \Box\varphi)$.

Secondly, $M, x \models \Box\varphi$. Indeed, if $x\, R\, z$ then $\widehat{x}\, r\, \widehat{z}$, so $\widehat{z} \in Y$ and $M, z \models \varphi$.

Now we use that $M \models \boxplus(\varphi \to \Box\varphi) \to (\Box\varphi \to \boxplus\varphi)$. Thus, $M, x \models \boxplus\varphi$. Recall that $x\, S\, y$. Then $M, y \models \varphi$, hence $\widehat{y} \in Y$. ☐

In Appendix (Lemma A.6) we strengthen the above lemma.

Now we come to the main technical tool of our paper.

Theorem 3.8 (Transfer of ADF to logics with transitive closure)
If the class $\mathsf{Mod}(L)$ admits definable filtration, then so does the class $\mathsf{Mod}(L^\boxplus)$.

Proof. IDEA:[4] in order to filtrate a model $M = (W, R, S, V) \models L^\boxplus$ for $\Gamma \subseteq \mathsf{Fm}(\Box, \boxplus)$, we build a special set of unimodal formulas Δ and Δ-filtrate the reduct $N = (W, R, V) \models L$ of M into a finite model $\widehat{N} = (\widehat{W}, \widehat{R}, \widehat{V}) \models L$. Then we show that $\widehat{N}^+ = (\widehat{W}, \widehat{R}, (\widehat{R})^+, \widehat{V}) \models L^\boxplus$ is a Γ-filtration of M. More precisely, we first take a modified valuation V^σ and actually filtrate N^σ, not N.

FORMALLY: take a model $M = (W, R, S, V)$ such that $M \models L^\boxplus$ and a finite Sub-closed set of formulas $\Gamma \subseteq \mathsf{Fm}(\Box, \boxplus)$. For each formula $\varphi \in \Gamma$, fix a fresh (not occurring in Γ) variable q_φ. Consider a substitution $\sigma \colon \mathsf{Var} \to \mathsf{Fm}(\Box, \boxplus)$ defined by $\sigma(q_\varphi) = \varphi$ for all $\varphi \in \Gamma$ and $\sigma(p) = p$ for all other variables p. In the definable variant $M^\sigma = (W, R, S, V^\sigma)$ of M we have: $M^\sigma \models q_\varphi \leftrightarrow \varphi$ for all $\varphi \in \Gamma$ (since $\varphi^\sigma = \varphi$), hence $M^\sigma \models \Box q_\varphi \leftrightarrow \Box\varphi$ and even $M^\sigma \models A \leftrightarrow A^\sigma$, for any formula $A \in \mathsf{Fm}(\Box)$. We also have $M^\sigma \models L^\boxplus$ by Lemma 3.6.

Now consider the reduct $N^\sigma = (W, R, V^\sigma)$ of M^σ. Clearly, $N^\sigma \models L$. However, we cannot Γ-filtrate this model, since Γ is a set of *bimodal* formulas.

---

[4] The proof of the main theorem differs from the proof of the corresponding Theorem 2.6 from our paper [12] in the following two aspects. First, in [12] we filtrate a model of the form $M = (W, R, R^+, V)$ such that $(W, R, R^+) \models L^\boxplus$, i.e., $(W, R) \models L$; while here we will filtrate a model of the form $M = (W, R, S, V)$ such that $M \models L^\boxplus$. As a consequence, in the old proof, we had to show that $(R^+)^{\mathsf{min}}_\sim \subseteq (R^{\mathsf{min}}_\sim)^+$, which is quite simple, while here we need to show that $\widehat{S} \subseteq (\widehat{R})^+$, for this we need Lemma 3.7. Secondly, we transform a filtration of (W, R, V) through a set of formulas $\Phi \subseteq \mathsf{Fm}(\Box)$ into a filtration of (W, R, S, V) through some set of formulas $\Phi' \subseteq \mathsf{Fm}(\Box, \boxplus)$, so we need to build Φ' from Φ.

Consider the following finite Sub-closed set of $\Box$-formulas:

$$\Delta \ := \ \{\, q_\varphi, \Box q_\varphi \ \mid \ \varphi \in \Gamma \,\} \quad \subset \quad \mathsf{Fm}(\Box).$$

$\mathsf{Mod}(L)$ admits definable filtration, so there is a Δ-filtration $\widehat{N^\sigma} = (\widehat{W}, \widehat{R}, \widehat{V^\sigma})$ of N^σ through some finite set $\Phi \subseteq \mathsf{Fm}(\Box)$ with $\Delta \subseteq \Phi$ such that $\widehat{N^\sigma} \models L$. Let us change $\widehat{V^\sigma}$ on the variables $p \in \Gamma$ by putting:[5] $\widehat{N^\sigma}, \widehat{x} \models p \leftrightharpoons \widehat{N^\sigma}, \widehat{x} \models q_p$.

Remark. Since we will have several models on the same set of points, we need a more subtle notation. In particular, we have $\widehat{W} = W/{\sim_\Phi^{N^\sigma}}$, this notation shows explicitly in which models we consider the $\sim_\Phi$-equivalence of points.

Now put $\widehat{M} := \widehat{N^\sigma}^{\oplus} = (\widehat{W}, \widehat{R}, (\widehat{R})^+, \widehat{V^\sigma})$. It remains to prove the following.

Claim. *The model $\widehat{M}$ is a Γ-filtration (through Φ^σ) of M.*

(1) We show that $\widehat{W} = W/{\sim_{\Phi^\sigma}^M}$. For any $x \in W$ and $A \in \mathsf{Fm}(\Box)$, we have:

$$N^\sigma, x \models A \quad \Leftrightarrow \quad M^\sigma, x \models A \quad \Leftrightarrow \quad M, x \models A^\sigma.$$

Therefore, for all $x, y \in W$, we have: $(x \sim_\Phi^{N^\sigma} y)$ iff $(x \sim_{\Phi^\sigma}^M y)$.
This allows us to introduce a simpler notation $\sim$ for $\sim_\Phi^{N^\sigma}$ and $\sim_{\Phi^\sigma}^M$.
Since $\widehat{N^\sigma}$ is a Δ-filtration of N^σ, we have: $R_\sim^{\mathsf{min}} \ \subseteq \ \widehat{R} \ \subseteq \ R_{\sim,\Delta}^{\mathsf{max}}$. $\quad (*)$

(2) The relation $\sim$ respects Γ. Indeed, $\Phi \supseteq \Delta \supseteq \{q_\varphi \mid \varphi \in \Gamma\}$, hence $\Phi^\sigma \supseteq \Gamma$.

(3) We show that $M, x \models p \Leftrightarrow \widehat{M}, \widehat{x} \models p$, for $p \in \Gamma$. Note that $\widehat{M} \models q_p \leftrightarrow p$.

$$\begin{array}{ccc} M, x \models p \ \Leftrightarrow \ M^\sigma, x \models q_p & \Leftrightarrow & N^\sigma, x \models q_p \\ & & \Updownarrow \\ \widehat{M}, \widehat{x} \models p \ \Leftrightarrow \ \widehat{M}, \widehat{x} \models q_p & \Leftrightarrow & \widehat{N^\sigma}, \widehat{x} \models q_p \end{array}$$

(4) $R_\sim^{\mathsf{min}} \ \subseteq \ \widehat{R}$. This holds by $(*)$.

(5) $S_\sim^{\mathsf{min}} \ \subseteq \ \widehat{S}$, where $\widehat{S} := (\widehat{R})^+$. Using (4) and Lemma 3.7, we obtain:

$$S_\sim^{\mathsf{min}} \ \subseteq \ (R_\sim^{\mathsf{min}})^+ \ \subseteq \ (\widehat{R})^+ \ = \ \widehat{S}.$$

(6) $\widehat{R} \subseteq R_{\sim,\Gamma}^{\mathsf{max}}$. Due to $(*)$, it suffices to prove that $R_{\sim,\Delta}^{\mathsf{max}} \subseteq R_{\sim,\Gamma}^{\mathsf{max}}$.
Assume that $\widehat{x}\,(R_{\sim,\Delta}^{\mathsf{max}})\,\widehat{y}$. To show that $\widehat{x}\,(R_{\sim,\Gamma}^{\mathsf{max}})\,\widehat{y}$, take any $\Box\varphi \in \Gamma$. Then:

$$\begin{array}{ccccc} M, x \models \Box\varphi & \overset{(a)}{\Longleftrightarrow} & M^\sigma, x \models \Box q_\varphi & \overset{(b)}{\Longleftrightarrow} & N^\sigma, x \models \Box q_\varphi \\ & & & & \Downarrow (c) \\ M, y \models \varphi & \overset{(a)}{\Longleftrightarrow} & M^\sigma, y \models q_\varphi & \overset{(b)}{\Longleftrightarrow} & N^\sigma, y \models q_\varphi \end{array}$$

We used: (a) Lemma 3.5; (b) $q_\varphi, \Box q_\varphi \in \mathsf{Fm}(\Box)$; (c) $\Box q_\varphi \in \Delta$ and $\widehat{x}\,(R_{\sim,\Delta}^{\mathsf{max}})\,\widehat{y}$.

(7) $\widehat{S} \subseteq S_{\sim,\Gamma}^{\mathsf{max}}$. Due to $(*)$, it suffices to prove that $(R_{\sim,\Delta}^{\mathsf{max}})^+ \subseteq S_{\sim,\Gamma}^{\mathsf{max}}$.
Let us denote $r := R_{\sim,\Delta}^{\mathsf{max}}$ and $s := S_{\sim,\Gamma}^{\mathsf{max}}$. In order to prove that $r^+ \subseteq s$, it suffices to prove two inclusions: $r \subseteq s$ and $r \circ s \subseteq s$.

(7a) Proof of $(r \subseteq s)$. We will use the axiom (A1): $\boxplus p \to \Box p$.
Assume $\widehat{x}\,(R_{\sim,\Delta}^{\mathsf{max}})\,\widehat{y}$. To prove that $\widehat{x}\,(S_{\sim,\Gamma}^{\mathsf{max}})\,\widehat{y}$, take any $\boxplus\varphi \in \Gamma$. Then:

[5] We could simply assume that $\widehat{V^\sigma}$ was undefined for the variables $p \in \mathsf{Var}(\Gamma)$ before we defined it here explicitly. This allows us to use the same notation $\widehat{V^\sigma}$ for the amended valuation. Note that this amendment does not change the truth of formulas from Δ in $\widehat{N^\sigma}$.

$$\begin{array}{ccccccc} M,x\models\boxplus\varphi & \overset{(d)}{\Longrightarrow} & M,x\models\Box\varphi & \overset{(a)}{\Longleftrightarrow} & M^\sigma,x\models\Box q_\varphi & \overset{(b)}{\Longleftrightarrow} & N^\sigma,x\models\Box q_\varphi \\ & & & & & & \Downarrow (c) \\ & & M,y\models\varphi & \overset{(a)}{\Longleftrightarrow} & M^\sigma,y\models q_\varphi & \overset{(b)}{\Longleftrightarrow} & N^\sigma,y\models q_\varphi \end{array}$$

(d) holds since $M\models\boxplus\varphi\to\Box\varphi$. The explanations of (a,b,c) are the same.

(7b) Proof of $(r\circ s\subseteq s)$. We will use the axiom (A2): $\boxplus p\to\Box\boxplus p$.
Assume $\widehat{x}\,(R^{\mathsf{max}}_{\sim,\Delta})\,\widehat{y}\,(S^{\mathsf{max}}_{\sim,\Gamma})\,\widehat{z}$. To prove $\widehat{x}\,(S^{\mathsf{max}}_{\sim,\Gamma})\,\widehat{z}$, take any $\boxplus\varphi\in\Gamma$. Then:

$$\begin{array}{ccccccc} M,x\models\boxplus\varphi & \overset{(e)}{\Longrightarrow} & M,x\models\Box\boxplus\varphi & \overset{(a)}{\Longleftrightarrow} & M^\sigma,x\models\Box q_{\boxplus\varphi} & \overset{(b)}{\Longleftrightarrow} & N^\sigma,x\models\Box q_{\boxplus\varphi} \\ & & & & & & \Downarrow (c) \\ M,z\models\varphi & \overset{(g)}{\Longleftarrow} & M,y\models\boxplus\varphi & \overset{(a)}{\Longleftrightarrow} & M^\sigma,y\models q_{\boxplus\varphi} & \overset{(b)}{\Longleftrightarrow} & N^\sigma,y\models q_{\boxplus\varphi} \end{array}$$

We used: (e) $M\models\boxplus\varphi\to\Box\boxplus\varphi$; (a) Lemma 3.5; (b) $\Box q_{\boxplus\varphi}\in\mathsf{Fm}(\Box)$; (c) $\Box q_{\boxplus\varphi}\in\Delta$ and $\widehat{x}\,(R^{\mathsf{max}}_{\sim,\Delta})\,\widehat{y}$; (g) $\boxplus\varphi\in\Gamma$ and $\widehat{y}\,(S^{\mathsf{max}}_{\sim,\Gamma})\,\widehat{z}$.

This completes the proof of theorem. □

Note that in (7a) and (7b) we proved inclusions that involve maximal relations, and these inclusions resemble the axioms (A1) and (A2). This is not a coincidence. In Lemma 4.3 of our paper [12], we already made this observation for any *right-linear grammar* axiom and both (A1) and (A2) are right-linear.

Let us summarize the main result on logics with transitive closure. We give two versions. The former theorem uses a rather unusual property (filtration of models). However, its advantage is that one can 'iterate' the application of this theorem (as we do in Section 4), since its premise and conclusion have the same form: "the class of models of a logic admits definable filtration". The latter theorem uses filtration of frames, but additionally requires canonicity.

Theorem 3.9 (Main result, version 1) *Assume that the class of models* $\mathsf{Mod}(L)$ *of a logic* L *admits definable filtration. Then:*

(1) *the class of models* $\mathsf{Mod}(L^{\boxplus})$ *admits definable filtration;*
(2) *hence the logic* $L^{\boxplus}$ *has the finite model property;*
(3) *hence the logic* $L^{\boxplus}$ *is Kripke complete.*

Proof. Assume $\mathsf{Mod}(L)$ admits definable filtration. Then so does $\mathsf{Mod}(L^{\boxplus})$, by Theorem 3.8. By Lemma 2.6, $L^{\boxplus}$ has the FMP and is Kripke complete. □

Theorem 3.10 (Main result, version 2) *Assume that a logic* L *is canonical and the class of its frames* $\mathsf{Fr}(L)$ *admits definable filtration. Then:*

(1) *the class* $\mathsf{Mod}(L^{\boxplus})$ *admits definable filtration;*
(2) *hence the logic* $L^{\boxplus}$ *has the finite model property;*
(3) *hence the logic* $L^{\boxplus}$ *is Kripke complete.*

Proof. If L is canonical and the class $\mathsf{Fr}(L)$ admits definable filtration, then so does the class $\mathsf{Mod}(L)$, by Theorem 2.10. Now apply Theorem 3.9. □

4 PDLization of logics that admit filtration

Now we apply Theorem 3.9 to show that if $\mathsf{Mod}(L)$ admits definable filtration, then the following PDL-like expansions of L have the finite model property.

Definition 4.1 For an alphabet Σ, let $\Sigma^\sharp = \Sigma \cup \{(e \circ f), (e \cup f), e^+ \mid e, f \in \Sigma\}$, assuming that the added symbols are not in Σ. Put $\Sigma^{(0)} = \Sigma$, $\Sigma^{(n+1)} = (\Sigma^{(n)})^\sharp$.

For a frame $F = (W, (R_e)_{e \in \Sigma})$, put $F^\sharp = (W, (R_e)_{e \in \Sigma^\sharp})$, where for $e, c \in \Sigma$,

$$R_{e \circ c} = R_e \circ R_c, \quad R_{e \cup c} = R_e \cup R_c, \quad R_{e^+} = (R_e)^+.$$

Put $F^{(0)} = F$, $F^{(n+1)} = (F^{(n)})^\sharp$.

For a model $M = (F, V)$, we put $M^\sharp = (F^\sharp, V)$ and $M^{(n)} = (F^{(n)}, V)$.

For a logic L over Σ, let $L^\sharp$ be the smallest (normal) logic over $\Sigma^\sharp$ that contains L and the following PDL-like axioms, for all $e, c \in \Sigma$:

$[e \cup c]p \leftrightarrow [e]p \wedge [c]p$,
$[e \circ c]p \leftrightarrow [e][c]p$,
$[e^+]p \to [e]p, \quad [e^+]p \to [e][e^+]p, \quad [e^+](p \to [e]p) \to ([e]p \to [e^+]p)$.

We put $L^{(0)} = L$, $L^{(n+1)} = (L^{(n)})^\sharp$.

The following is a simple analogue of Lemma 3.2.

Lemma 4.2 **(a)** *$M \models L$ implies $M^\sharp \models L^\sharp$.* **(b)** *$F \models L$ iff $F^\sharp \models L^\sharp$.*

By an easy induction on n, we obtain

Proposition 4.3 *For a frame F and $n < \omega$, $F \models L$ iff $F^{(n)} \models L^{(n)}$.*

Proposition 4.4 *For a logic L and $n < \omega$, $L^{(n)}$ is conservative over L.*

Proof. As in Lemma 3.3, using ${M_L}^{(n)}$ instead of $M_L^\oplus$ and Lemma 4.2(a). □

Lemma 4.5 *Let L be a logic over Σ, $e, c \in \Sigma$. Let L_1 and L_2 be the logics over $\Sigma \cup \{g\}$, where $g \notin \Sigma$, such that*

L_1 extends L with the axiom $[g]p \leftrightarrow [e]p \wedge [c]p$,
L_2 extends L with the axiom $[g]p \leftrightarrow [e][c]p$.

If $\mathsf{Mod}(L)$ admits definable filtration, then so do $\mathsf{Mod}(L_1)$ and $\mathsf{Mod}(L_2)$.

Proof. Straightforward. Details can be reconstructed from the proof of Lemma 2.3 in [12], which is the analog of our lemma for the classes of frames. □

Theorem 4.6 *Let L be a logic over a finite alphabet Σ. If the class of its models $\mathsf{Mod}(L)$ admits definable filtration, then, for every $n < \omega$, we have:*

(i) *$\mathsf{Mod}(L^{(n)})$ admits definable filtration.*
(ii) *$L^{(n)}$ has the finite model property; a fortiori, $L^{(n)}$ is Kripke complete.*
(iii) *If L is finitely axiomatizable, then $L^{(n)}$ is decidable.*
(iv) *If the class of finite frames of L is decidable, then $L^{(n)}$ is co-recursively enumerable.*

Proof. (i) By Theorem 3.8 and Lemma 4.5, if $\mathsf{Mod}(L)$ admits definable filtration, then so does $\mathsf{Mod}(L^\sharp)$. So, (i) follows by induction on n.

(ii) By Lemma 2.6.

(iii) Note that if L is finitely axiomatizable, then so is $L^{(n)}$. The claim then follows from Harrop's Theorem (see Section 1).

(iv) If the class of finite frames of L is decidable, then the class of finite frames of $L^{(n)}$ is decidable, too. In this case $L^{(n)}$ is co-recursively enumerable, since $L^{(n)}$ is the logic of its finite frames. □

Theorem 4.6 can be generalized for the case when we additionally extend the alphabet with converse modalities. This generalization can be obtained by modifying the proof of Theorem 2.6 in [12].

4.1 Fusions that admit filtration

Here we consider a special kind of definable filtration, called *strict filtration*.

Definition 4.7 If, in terms of Definition 2.1, $\sim = \sim_\Gamma$, then we call the filtration $\widehat{M}$ *strict*. The corresponding notions "a class of frames (or models) *admits strict filtration*" are introduced in the obvious way.

Strict filtration is the most standard variant of filtration; it is well-known that the classes of frames of the logics $\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}$ admit strict filtration (for the logics **K** and **T**, even the *minimal* strict filtration works; for **K4**, **S4**, **S5**, strict filtration is obtained by taking the transitive closure of the minimal filtered relation [15]).

Let us recall the notion of the *fusion* of logics. Let $L_1, \ldots, L_k$ be logics over finite alphabets $\Sigma_1, \ldots, \Sigma_k$. Without loss of generality we assume that these alphabets are disjoint. The *fusion* $L_1 * \ldots * L_k$ of these logics is the smallest normal logic over the alphabet $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_k$ that contains $L_1 \cup \ldots \cup L_k$.

It is well-known that the fusion operation preserves Kripke completeness, the finite model property, and decidability [14]. We observe that it also preserves the property "a logic admits strict filtration".

Theorem 4.8 (Fusion and strict filtration) *If classes of frames* $\mathsf{Fr}(L_i)$, $1 \le i \le k$, *admit strict filtration, then* $\mathsf{Fr}(L_1 * \ldots * L_k)$ *admits strict filtration.*

Proof. The idea is the same as in the proof of Theorem 3.8. To simplify notation, we consider the case of unimodal logics. Let $L = L_1 * \ldots * L_k$, $M = (F, V)$ be a model on an L-frame $F = (W, R_1, \ldots, R_k)$, $\Gamma \subseteq \mathsf{Fm}(\Box_1, \ldots, \Box_k)$ be finite and Sub-closed. For $\varphi \in \Gamma$, we take fresh variables q_φ, and consider a model $M' = (F, V')$ such that

$$M, x \models \varphi \;\text{ iff }\; M', x \models \varphi \;\text{ iff }\; M', x \models q_\varphi$$

for all x in M. For $1 \le i \le k$, we put:

$$\Gamma_i = \{q_\varphi \mid \varphi \in \Gamma\} \cup \{\Box q_\varphi \mid \Box_i \varphi \in \Gamma\}.$$

Note that $\Gamma_i \subseteq \mathsf{Fm}(\Box)$. Let $\sim_i$ be the equivalence induced by Γ_i in the model $M_i = (W, R_i, V')$, and $\sim_\Gamma$ the equivalence induced by Γ in M. Observe that

$$M_i, x \models \Box q_\varphi \;\text{ iff }\; M, x \models \Box_i \varphi \;\text{ for all }\; \varphi \in \Gamma. \qquad (*)$$

Therefore, one can see that $\sim_i = \sim_\Gamma$ for all i. Put $\widehat{W} = W/{\sim_\Gamma}$. For each i, there exists a filtration $\widehat{M}_i = (\widehat{W}, \widehat{R}_i, \widehat{V}_i)$ of M_i through Γ_i such that $(\widehat{W}, \widehat{R}_i) \models L_i$. The valuations $\widehat{V}_i$ coincide on the variables q_φ. W.l.o.g., they also coincide on other variables (since they do not occur in Γ_i), and that $\widehat{M}_i, \widehat{x} \models p$ iff $M, x \models p$ for each variable $p \in \Gamma$. The resulting valuation on $\widehat{W}$ is denoted by $\widehat{V}$.

Consider the model $\widehat{M} = (\widehat{W}, \widehat{R_1}, \dots, \widehat{R_k}, \widehat{V})$. Note that its frame validates the fusion L. We claim that $\widehat{M}$ is a filtration of M through Γ. Clearly, $\widehat{R}_i$ contains the i-th minimal filtered relation. To check that $\widehat{R_i}$ is contained in the i-th maximal filtered relation, assume that $\widehat{x}\widehat{R_i}\widehat{y}$, $M, x \models \Box_i\varphi$, and $\Box_i\varphi \in \Gamma$. Then $M_i, x \models \Box q_\varphi$, by (*). Since $\widehat{M_i}$ is a filtration of M_i through Γ_i and $\Box q_\varphi \in \Gamma_i$, we have $\widehat{M_i, \widehat{y}} \models q_\varphi$. By Filtration lemma, $M_i, y \models q_\varphi$. Hence, $M', y \models q_\varphi$ and we conclude that $M, y \models \varphi$, as required. $\Box$

Theorem 4.9 *Let $L_1, \dots, L_k$ be canonical logics and their classes of frames $\mathsf{Fr}(L_i)$, $1 \le i \le k$, admit strict filtration. Then, for every $n < \omega$, the logic $(L_1 * \dots * L_k)^{(n)}$ has the finite model property.*

Proof. The fusion $L = L_1 * \dots * L_k$ is canonical. By Theorem 4.8, the class $\mathsf{Fr}(L)$ admits strict filtration. Hence $\mathsf{Mod}(L)$ admits definable filtration, by Theorem 2.10. Finally, $(L)^{(n)}$ has the FMP, by Theorem 4.6. $\Box$

4.2 A class of formulas that admit strict filtration

We present a collection of modal formulas that admit strict (and so definable) filtration. The obvious candidates are modal formulas whose first-order equivalents belong to a certain FO fragment we call MFP.[6] We define it inductively as the minimal set of FO formulas satisfying the following conditions:

- if x and y are variables, R is a binary relation symbol, then $R(x, y) \in \mathsf{MFP}$ and $x = y \in \mathsf{MFP}$;
- if A and B are in MFP, then $(A \land B)$ and $(A \lor B)$ are in MFP;
- if $A \in \mathsf{MFP}$, and v is a variable, then $\forall v\, A$ and $\exists v\, A$ are in MFP;
- if x and y are variables, R is a binary relation symbol, and $A \in \mathsf{MFP}$, then $\forall x \forall y (R(x, y) \to A)$ and $\forall x \forall y (x = y \to A)$ are in MFP.

This definition is the restriction of the fragment $\mathsf{POS} + \forall\mathsf{G}$ from [9] to the first-order language with only binary predicates. Examples of MFP-sentences are reflexivity $\forall x R(x, x)$, symmetry $\forall x \forall y (R(x, y) \to R(y, x))$, and density $\forall x \forall y (R(x, y) \to \exists z\, (R(x, z) \land R(z, y)))$, but not transitivity.

FO counterparts of minimal filtrations are strong onto homomorphisms.

Definition 4.10 Given two frames $F = (W, R)$ and $F' = (W', R')$, a map $h \colon W \to W'$ is a *strong onto homomorphism* if h is onto and we have:

- for all $x, y \in W$, if $x\, R\, y$, then $h(x)\, R'\, h(y)$ (*monotonicity*);
- for all $x', y' \in W'$, if $x'\, R'\, y'$, then there exist $x, y \in W$ such that $h(x) = x'$, $h(y) = y'$, and $x\, R\, y$ (*weak lifting*).

Note that a strong homomorphism h from F onto F' induces an equivalence $\sim$ on W defined by $x \sim y$ iff $h(x) = h(y)$, and then F' is isomorphic to the *minimal filtrated frame* $F_\sim^{\mathsf{min}} = (W/{\sim}, R_\sim^{\mathsf{min}})$. Conversely, if $\widehat{M}$ is a minimal filtration of M, then the map $x \mapsto \widehat{x}$ is a strong homomorphism from F onto $\widehat{F}$.

[6] The abbreviation stems from "*p*reserved under *m*inimal *f*iltration".

Any MFP-formula is preserved under strong onto homomorphisms [9, Prop. 5.2]. Moreover, any FO formula with binary relations that is preserved under strong onto homomorphisms is equivalent to some MFP-formula [11].

Definition 4.11 A modal formula φ is called a *modal* MFP*-formula* if it has a FO equivalent (on frames) in MFP.

Typical examples of modal MFP-formulas are expressions of the form $p \wedge \Diamond q \to \psi$, where ψ is a positive modal formula. Note that these examples are Sahlqvist formulas, and hence canonical.

Theorem 4.12 *For any set Φ of modal* MFP*-formulas over a finite alphabet Σ, the class of frames $\mathsf{Fr}(\mathbf{K}_\Sigma + \Phi)$ admits strict filtration.*

Proof. Denote $\mathcal{F} = \mathsf{Fr}(\mathbf{K}_\Sigma + \Phi)$. Let $M = (F, V)$ be an $\mathcal{F}$-model and Γ a finite Sub-closed set of formulas. Take the minimal filtration $\widehat{M} = (\widehat{F}, \widehat{V})$ of M through Γ; note that this filtration is strict. Then the map $x \mapsto \widehat{x}$ is a strong homomorphism from F onto $\widehat{F}$. Since the set Φ^* of the MFP first-order equivalents of Φ is true in F, it is also true in $\widehat{F}$. Hence $\widehat{M}$ is an $\mathcal{F}$-model. $\Box$

From Theorem 4.9, we obtain:

Corollary 4.13 *Let each $L_1, \ldots, L_k$ be any of the logics $\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}$, or a logic axiomatized by canonical modal* MFP*-formulas. Then, for any $n < \omega$, the logic $(L_1 * \ldots * L_k)^{(n)}$ has the finite model property.*

5 Conclusions and further research

We proved that if L is a canonical logic, and the class of its frames $\mathsf{Fr}(L)$ admits definable filtration, then the logic $L^{\boxplus}$ is Kripke complete and, moreover, has the FMP (and is decidable, if L was finitely axiomatizable). The first problem we pose is whether we can weaken the pre-conditions and obtain the completeness of $L^{\boxplus}$ without obtaining the FMP.

Problem 1. *If a logic L is canonical, then is the logic $L^{\boxplus}$ Kripke complete?*

Next, can we weaken the 'canonicity' to the 'completeness' in Theorem 3.10?

Problem 2. *If a logic L is complete and the class of its frames $\mathsf{Fr}(L)$ admits definable filtration, then does the same hold for the logic $L^{\boxplus}$?*

The following questions are of more technical character.

Question 1. Is it the case that whenever the class of models (or frames) of a logic L admits filtration, it also admits definable filtration?

Question 2. Let us replace the axiom (A2) $\boxplus p \to \Box\boxplus p$ with (A2′) $\boxplus p \to \boxplus\Box p$ in the logic $\mathbf{K}^{\boxplus}$. Do we obtain the same logic, i.e., does this logic derive (A2)? Note that the frames for it are the same as for $\mathbf{K}^{\boxplus}$, see Lemma A.4(6).

Question 3. Is the logic $\mathbf{K.2}^{\boxplus}$ Kripke complete? (We conjecture: yes.)

Recall that the logic $\mathbf{K.2}$ extends $\mathbf{K}$ with the formula $\Diamond\Box p \to \Box\Diamond p$. It is canonical and hence complete with respect to the class of frames (W, R) that satisfy the first-order *convergence* (or Church–Rosser) condition:

$$\forall x, y, z\, (x\, R\, y \;\wedge x\, R\, z \;\Rightarrow\; \exists w\, (y\, R\, w \;\wedge z\, R\, w)).$$

Our main result is not applicable to this logic, since the class of its frames $\mathsf{Fr}(\mathbf{K.2})$ does not admit filtration, as we established in [12, Theorem 5.4].

If R is convergent, then so is R^+ (easy exercise). Is modal logic able to establish this? That is, can we derive the formula $\diamondplus\boxplus p \to \boxplus\diamondplus p$ in $\mathbf{K.2}^{\boxplus}$? We succeeded in deriving it (see Lemma A.7 in Appendix).

Question 3. In Lemma A.6, the bimodal formula $\boxplus(p \to \Box p) \to (\Box p \to \boxplus p)$ is shown to have the following property crucial for our main result: *if all its substitution instances are true in some model $M = (F, V)$, then this formula is valid on the frame of every definable minimal filtration: if $M \models A^*$ then $F^{\mathsf{min}}_{\sim\Phi} \models A$, for any finite set of formulas Φ.* Are there any other examples of such formulas? How is this property related to the admissibility of filtration, completeness, decidability of a logic axiomatized by such formulas?

Appendix

A.1 On modally differentiated filtration

Any Γ-filtration $\widehat{M}$ of a model M through the same set Γ, i.e., through $\sim_\Gamma$, is always differentiated: indeed, if $[x]_{\sim_\Gamma} \neq [y]_{\sim_\Gamma}$, then the points x and y in M differ by some formula $\varphi \in \Gamma$; by Filtration Lemma 2.2, the truth of all formulas from Γ is preserved, so the points $\widehat{x}$ and $\widehat{y}$ differ by the same formula φ.

On the contrary, a Γ-filtration $\widehat{M}$ of M through some set $\Phi \supseteq \Gamma$ is not necessarily differentiated: in the above argument, x and y will differ by some $\varphi \in \Phi$, and the Filtration Lemma transfers the truth of formulas from Γ only.

Lemma A.2 below resolves this obstacle: by possibly changing the set Φ, a filtration can be made differentiated. We will need the following simple fact.

Proposition A.1 *Let M be a model and $\sim$ an equivalence relation on W of finite index. Then $\sim$ is of the form $\sim_\Phi$, for some finite set of formulas Φ, iff each equivalence class $[x]_\sim$ is defined in M by some formula.*

Proof. If Φ is finite, then every class $[x]_{\sim_\Phi} \subseteq W$ is defined by the formula

$$\bigwedge\bigl(\{\varphi \mid \varphi \in \Phi \text{ and } M, x \models \varphi\} \cup \{\neg\varphi \mid \varphi \in \Phi \text{ and } M, x \models \neg\varphi\}\bigr).$$

Conversely, if $\sim$ partitions W into finitely many classes and each class is defined by a formula φ_i, $1 \le i \le n$, then clearly $\sim\, = \,\sim_\Phi$ for $\Phi = \{\varphi_1, \dots, \varphi_n\}$. □

Lemma A.2 *Assume that $\mathsf{Mod}(L)$ admits (definable) filtration. Then for every finite Sub-closed set of formulas Γ and every model $M \in \mathsf{Mod}(L)$, there exists a (definable) Γ-filtration $\widehat{M} \in \mathcal{M}$ of M that is a differentiated model.*

Proof. Idea: first, build a Γ-filtration M_1 of M, then an Fm-filtration M_2 of M_1; finally, build a differentiated Γ-filtration $\widehat{M}$ of M that is isomorphic to M_2.

Formally, let $M = (W, R, V)$, $M \models L$, and let Γ be as stated above.

(1) Since $\mathsf{Mod}(L)$ admits filtration, there is a Γ-filtration $M_1 = (W_1, R_1, V_1)$ of M with $M_1 \models L$. So, $W_1 = W/{\sim}$ for some equivalence relation $\approx$ of finite

index, $\sim$ respects Γ, $R^{\mathsf{min}}_{\approx} \subseteq R_1 \subseteq R^{\mathsf{max}}_{\approx,\Gamma}$, V_1 is defined canonically on $\mathsf{Var}(\Gamma)$.

(2) Let $M_2 = (W_2, R_2, V_2)$ be a filtration of M_1 through the set of all formulas.[7] So, $W_2 = W_1/\equiv$, where $\equiv$ is the *modal equivalence* relation; V_2 is canonical on all variables. By the Filtration lemma 2.2, $M_1 \equiv M_2$, so $M_2 \models L$.

(3) Now we build a model $\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{V})$ isomorphic to M_2 as follows. Put $\widehat{W} := W/\sim$, where, for all $x, y \in W$, we define an equivalence relation $\sim$ by

$$x \sim y \quad \overset{\text{def}}{\iff} \quad (M_1, [x]_\approx) \equiv (M_1, [y]_\approx) \quad \iff \quad [[x]_\approx]_\equiv = [[y]_\approx]_\equiv .$$

Claim 1. *The function $h([x]_\sim) = [[x]_\approx]_\equiv$ is a bijection between $\widehat{W}$ and W_2.*

Proof. Easy. This does not rely on the fact that $\approx$ and $\equiv$ are of finite index.

From now on, we denote $\widehat{x} = [x]_\sim$.

Claim 2. *The equivalence relation $\sim$ on W respects Γ: if $x \sim y$, then $x \sim_\Gamma y$.*

Proof. If $x, y \in W$ and $x \sim y$ then, by the Filtration lemma 2.2, we have:

$$(M, x) \quad \sim_\Gamma \quad (M_1, [x]_\approx) \quad \sim_{\mathsf{Fm}} \quad (M_1, [y]_\approx) \quad \sim_\Gamma \quad (M, y).$$

Using the bijection h, we transfer R_2 and V_2 to $\widehat{M}$ in the obvious way:

$$\widehat{x}\,\widehat{R}\,\widehat{y} \;\overset{\text{def}}{\iff}\; h(\widehat{x})\,R_2\,h(\widehat{y}); \qquad \widehat{x} \models q \;\overset{\text{def}}{\iff}\; M_2, h(\widehat{x}) \models q, \text{ for all } q \in \mathsf{Var}.$$

Since the models $\widehat{M}$ and M_2 are isomorphic, we have $\widehat{M} \models L$.

Claim 3. *$\widehat{V}$ is canonical on each $p \in \mathsf{Var}(\Gamma)$: $M, x \models p \;\Leftrightarrow\; \widehat{M}, \widehat{x} \models p$.*

Proof. Indeed: $(M, x) \sim_\Gamma (M_1, [x]_\approx) \sim_{\mathsf{Fm}} (M_2, [[x]_\approx]_\equiv) \sim_{\mathsf{Var}} (\widehat{M}, \widehat{x})$.

Claim 4. *The inclusions $R^{\mathsf{min}}_{\sim} \subseteq \widehat{R} \subseteq R^{\mathsf{max}}_{\sim,\Gamma}$ hold.*

Proof. (min) Clearly, $x\,R\,y \Rightarrow [x]_\approx\,R_1\,[y]_\approx \Rightarrow [[x]_\approx]_\equiv\,R_2\,[[y]_\approx]_\equiv \Leftrightarrow \widehat{x}\,\widehat{R}\,\widehat{y}$.
(max) If $\widehat{x}\,\widehat{R}\,\widehat{y}$, then $[[x]_\approx]_\equiv\,R_2\,[[y]_\approx]_\equiv$. But $R_2 \subseteq (R_1)^{\mathsf{max}}_{\equiv,\mathsf{Fm}}$. So, for $\Box A \in \Gamma$,
$M, x \models \Box A \Leftrightarrow M_1, [x]_\approx \models \Box A \Rightarrow M_1, [y]_\approx \models A \Leftrightarrow M, y \models A$.

Claim 5. *If M_1 is a definable filtration of M, then $\widehat{M}$ is definable too.*

Proof. Assume M_1 is a filtration of M through a finite Φ. By Proposition A.1, each $\sim_\Phi$-class is defined by some formula φ_i. Each $\sim$-class is the *union* of some $\sim_\Phi$-classes (namely, those that are modally equivalent as points in M_1). Hence, each $\sim$-class is defined by the *disjunction* of some formulas φ_i. By Proposition A.1, $\sim = \sim_\Psi$, for some set of formulas Ψ, thus $\widehat{M}$ is definable. $\Box$

A.2 On filtration of the canonical model of a theory of a model

Lemma A.3 (Filtration and canonical mapping) *Let $M = (W, R, V)$ be a model, $M_T = (W_T, R_T, V_T)$ the canonical model of its theory $T = \mathsf{Th}(M)$, and $t\colon M \to M_T$ the canonical mapping: $t(a) = \mathsf{Th}(M, a) \in W_T$, for $a \in W$.*

Then, for any finite set of formulas Φ, we have a bijection between the (finite) quotient sets $W/{\sim_\Phi}$ and $W_T/{\sim_\Phi}$ defined, for $a \in W$, by

$$f([a]_{\sim_\Phi}) \;:=\; [t(a)]_{\sim_\Phi}.$$

Proof. We denote $\widehat{a} := [a]_{\sim_\Phi}$. Note that $\widehat{x} = \widehat{y}$ iff $x \cap \Phi = y \cap \Phi$, for all

[7] In fact, if a filtration through the set of all formulas is finite, then it is unique, i.e., the minimal and the maximal relations coincide. But here we do not need this fact.

$x, y \in W_T$. Hence, by definition of f, for all $a \in W$ and $x \in W_T$, we have

$$f(\widehat{a}) = \widehat{x} \iff t(a) \cap \Phi = x \cap \Phi.$$

First, let us show that f is well-defined and injective: for all $a, b \in W$:

$$\widehat{a} = \widehat{b} \Leftrightarrow a \sim_\Phi b \Leftrightarrow \mathsf{Th}(M, a) \cap \Phi = \mathsf{Th}(M, b) \cap \Phi \Leftrightarrow [t(a)]_{\sim_\Phi} = [t(b)]_{\sim_\Phi}.$$

To prove that f is surjective, take any $\widehat{x} \in (W_T/{\sim_\Phi})$. Denote $A := \bigwedge(x \cap \Phi')$, where $\Phi' = \Phi \cup \{\neg B \mid B \in \Phi\}$. Clearly, $A \in x$. Now $M \not\models \neg A$, for otherwise $\neg A \in \mathsf{Th}(M) = T \subseteq x$ and x is inconsistent.

Thus, A is satisfiable in M, so $M, a \models A$ for some $a \in W$. We claim that $f(\widehat{a}) = \widehat{x}$, i.e., for all $B \in \Phi$, we have $M, a \models B$ iff $B \in x$. If $B \in x$, then $B \in (x \cap \Phi')$, so $M, a \models B$. If $B \notin x$, then $\neg B \in (x \cap \Phi')$, so $M, a \models \neg B$. □

A.3 On the semantics of Segerberg's axioms

For convenience, let us recall the axioms for the transitive closure modality:

$$(\mathsf{A1})\ \boxplus p \to \Box p, \qquad (\mathsf{A2})\ \boxplus p \to \Box\boxplus p, \qquad (\mathsf{A3})\ \boxplus(p \to \Box p) \to (\Box p \to \boxplus p).$$

We will also consider the following modified axiom: $(\mathsf{A2'})\ \boxplus p \to \boxplus\Box p$.

Lemma A.4 *Let $F = (W, R, S)$ be a bi-modal frame.*

(1) $F \models (\mathsf{A1}) \iff S \supseteq R$.
(2) $F \models (\mathsf{A2}) \iff S \supseteq R \circ S$.
(3) $F \models (\mathsf{A1}) \wedge (\mathsf{A2}) \implies S \supseteq R^+$; *the converse does not hold in general.*
(4) $F \models (\mathsf{A3}) \implies S \subseteq R^+$; *the converse does not hold in general.*
(5) $F \models (\mathsf{A1}) \wedge (\mathsf{A2}) \wedge (\mathsf{A3}) \iff S = R^+$.
(6) $F \models (\mathsf{A1}) \wedge (\mathsf{A2'}) \wedge (\mathsf{A3}) \iff S = R^+$.

Proof. The facts (1) and (2) are well-known. They imply $S \supseteq R^n$, for all $n \geq 1$, and thus (3) follows. Also (3) and (4) imply (5,⇒). So it remains to prove (4), (5,⇐), and (6) and provide counterexamples to (3,⇐) and (4,⇐).

(4,⇒) Assume $F \models (\mathsf{A3})$ and xSy; we need to prove that xR^+y. Denote $P = R^+(x) \subseteq W$; we need to show that $y \in P$. Consider a model $M = (F, V)$ with the valuation $V(p) = P$. Clearly, $M, x \models \Box p$, since $R(x) \subseteq R^+(x) = P$. Next, $M \models p \to \Box p$, since $P \supseteq R(P)$. Hence $M, x \models \boxplus(p \to \Box p)$. But $M, x \models (\mathsf{A3})$. Hence $M, x \models \boxplus p$. Since xSy, we obtain $M, y \models p$ and so $y \in V(p) = P$.

(5,⇐) Suppose $S = R^+$. Clearly, $S \supseteq R$ and $S \supseteq R \circ S$, hence $F \models (\mathsf{A1}) \wedge (\mathsf{A2})$, by (1) and (2). To prove that $F \models (\mathsf{A3})$, take any model $M = (F, V)$ and $x \in W$. Assume that $x \models \boxplus(p \to \Box p)$ and $x \models \Box p$. We need to show that $x \models \boxplus p$, i.e., $y \models p$ for all $y \in S(x)$. Recall that $S = R^+ = \bigcup_{n \geq 1} R^n$. Therefore, it remains to show, for every $n \geq 1$, that $y \models p$ for all $y \in R^n(x)$. We do this by induction. Induction base ($n = 1$) holds since $x \models \Box p$. Induction step: assume $x\, R^{n+1} y$, hence $x\, R^n t\, R\, y$ for some t. By induction hypothesis, $t \models p$. Since $S \supseteq R^+$, we have $S \supseteq R^n$. Thus $x\, S\, t$. Recall that $x \models \boxplus(p \to \Box p)$. Then $t \models p \to \Box p$, whence $t \models \Box p$ and $y \models p$.

(6) Clearly, $F \models (\mathsf{A2'})$ iff $S \supseteq S \circ R$. So, $F \models (\mathsf{A1}) \wedge (\mathsf{A2'})$ implies $S \supseteq R^+$, and $F \models (\mathsf{A3})$ implies $S \subseteq R^+$. Thus (6,⇒) is proved. The implication (6,⇐) is easy, since $S = R^+$ implies $S \supseteq S \circ R$, and so $F \models (\mathsf{A1}) \wedge (\mathsf{A2'}) \wedge (\mathsf{A3})$.

Here is a counterexample $M = (W, R, S, V)$ to (4,$\Leftarrow$): $W = \{a, b, c\}$, $aRbRc$ (R is not transitive), aSc, $V(p) = \{1\}$. Clearly, $S \supset R^+$. But $M, a \not\models (\mathsf{A3})$.

To refute (3,$\Leftarrow$), we show that $S \supseteq R^+$ does not imply $S \supseteq R \circ S$. Take $W = \{a, b\}$, aRb, aSb, bSa. Then $R^+ = R \subseteq S$. But $a(R{\circ}S)a$ and $\neg(aSa)$. $\Box$

A.4 On induction axiom and minimal filtrated frame

Lemma A.5 *The formula* $(\mathsf{A3})$ $\boxplus(p \to \Box p) \to (\Box p \to \boxplus p)$ *is not canonical.*

Proof. Denote $L = \mathbf{K}(\Box, \boxplus) \oplus (\mathsf{A3})$ and its canonical frame $F_L = (W, R, S)$. By Lemma A.4(4), to prove that $F \not\models (\mathsf{A3})$, it suffices to show that $S \not\subseteq R^+$.

The set $\Gamma = \{\neg\boxplus p\} \cup \{\Box^n p \mid n \geq 1\}$ is L-consistent, because every finite set of the form $\{\neg\boxplus p, \Box p, \ldots, \Box^n p\}$ is L-satisfiable (in a chain of length $n+1$). Hence $\Gamma \subseteq x$, for some maximal L-consistent set $x \in W$. Since $\neg\boxplus p \in x$, we have $M_L, x \not\models \boxplus p$ (later, we omit M_L). Hence, for some $y \in W$, we have $x\,S\,y$ and $y \models p$. However, $\neg(x\,R^+\,y)$; indeed, otherwise $x\,R^n\,y$, for some $n \geq 1$, and since $\Box^n p \in x$, we obtain $x \models \Box^n p$ and $y \models p$, a contradiction. $\Box$

We could prove the same using variable-free formulas only: put $p := \Diamond\top$.

Let us strengthen Lemma 3.7 (recall that $G \models (\mathsf{A3})$ implies $S \subseteq R^+$). Denote the *minimal filtered* (through Φ) frame by $G^{\mathsf{min}}_{\sim_\Phi} = (W/{\sim_\Phi}, R^{\mathsf{min}}_{\sim_\Phi}, S^{\mathsf{min}}_{\sim_\Phi})$.

Lemma A.6 (Induction axiom and minimal filtrated frame) *Let* $M = (W, R, S, V) \models (\mathsf{A3})^*$ *and let* $\Phi \subseteq \mathsf{Fm}$ *be finite. Then* $G^{\mathsf{min}}_{\sim_\Phi} \models (\mathsf{A3})$.

Proof. The minimal filtration model $\widehat{M} := M^{\mathsf{min}}_{\sim_\Phi} = (G^{\mathsf{min}}_{\sim_\Phi}, \widehat{V})$ is a Φ-filtration of M through Φ, hence it is a finite *differentiated* model (see Section A.1).

Due to Lemma 1.1, in order to prove our lemma, it suffices to show that

$$M \models (\mathsf{A3})^* \quad \text{implies} \quad \widehat{M} := M^{\mathsf{min}}_{\sim_\Phi} \models (\mathsf{A3})^*.$$

Assume $\widehat{M} \not\models (\mathsf{A3})[p := B]$, for some formula B. Then there is $\widehat{x} \in \widehat{W}$ such that **(a)** $\widehat{x} \models \boxplus(B \to \Box B)$, **(b)** $\widehat{x} \models \Box B$, **(c)** $\widehat{x} \not\models \boxplus B$. Hence there is $\widehat{y} \in \widehat{W}$ such that $\widehat{x}\,\widehat{S}\,\widehat{y}$ and **(d)** $\widehat{y} \not\models B$. Since $\widehat{x}\,S^{\mathsf{min}}_{\sim_\Phi}\,\widehat{y}$, without loss of generality, $x\,S\,y$.

Consider $Y := \widehat{V}(B) = \{\widehat{z} \in \widehat{W} \mid \widehat{M}, \widehat{z} \models B\}$. As in Lemma 3.7, Y is a finite collection of definable subsets of W, hence their union $\bigcup Y$ is also a definable subset of W. So, there is a formula φ such that, for all $z \in W$, we have:

$$M, z \models \varphi \;\Leftrightarrow\; z \in \bigcup Y \;\Leftrightarrow\; \widehat{z} \in Y = \widehat{V}(B) \;\Leftrightarrow\; \widehat{M}, \widehat{z} \models B.$$

Now we show that $M, x \not\models (\mathsf{A3})[p := \varphi]$, in contradiction with $M \models (\mathsf{A3})^*$.

(a') $M, x \models \boxplus(\varphi \to \Box\varphi)$. Indeed, take any $a, b \in W$ such that $x\,S\,a\,R\,b$ and $a \models \varphi$. Then $\widehat{x}\,\widehat{S}\,\widehat{a}\,\widehat{R}\,\widehat{b}$ and $\widehat{a} \models B$. Hence $\widehat{b} \models B$ by **(a)**, and so $b \models \varphi$.

(b') $M, x \models \Box\varphi$. Indeed, if $x\,R\,z$, then $\widehat{x}\,\widehat{R}\,\widehat{z}$; hence $\widehat{z} \models B$ by **(b)**, so $z \models \varphi$.

(d') $M, x \not\models \boxplus\varphi$. Indeed, $x\,S\,y$ and $M, y \not\models \varphi$, because $\widehat{y} \not\models B$ by **(d)**. $\Box$

A.5 On the logic of convergent frames

For convenience, we repeat the axioms for the transitive closure modality:

(A1) $\boxplus p \to \Box p$, (A2) $\boxplus p \to \Box\boxplus p$, (A3) $\boxplus(p \to \Box p) \to (\Box p \to \boxplus p)$.

Note that in any logic $L^{\boxplus}$, the following inference rule is derivable:

$$\frac{\varphi \to \Box\varphi}{\varphi \to \boxplus\varphi} \tag{R$\boxplus$}$$

Indeed, here is a derivation:
1) $\varphi \to \Box\varphi$. 2) $\boxplus(\varphi \to \Box\varphi)$. 3) $\Box\varphi \to \boxplus\varphi$ by (A3). 4) $\varphi \to \boxplus\varphi$ from 1 and 3.

Furthermore, in any logic $L^{\boxplus}$, the following formula is derivable:

$$\Box p \ \wedge \ \boxplus\Box p \ \to \ \boxplus p \tag{A$\boxplus$}$$

since one of its premises, $\boxplus\Box p$, is stronger then the premise $\boxplus(p \to \Box p)$ in (A3).

Recall that the logic **K.2** extends **K** with the axiom $\Diamond\Box p \to \Box\Diamond p$.

Lemma A.7 (Convergence for transitive closure) $\mathbf{K.2}^{\boxplus} \vdash \diamondplus\boxplus p \to \boxplus\diamondplus p$.

Proof. The proof is in two stages.

(1) We derive $\Diamond\boxplus p \longrightarrow \boxplus\Diamond p$, using $\Diamond\Box\varphi \to \Box\Diamond\varphi$ for $\varphi = p$ and $\varphi = \boxplus p$:

$$\begin{array}{llll}
\Diamond\boxplus p \xrightarrow{(A1)} \Diamond\Box p & \xrightarrow{.2} \Box\Diamond p. & (a) \\
\Diamond\boxplus p \xrightarrow{(A2)} \Diamond\Box\boxplus p & \xrightarrow{.2} \Box\Diamond\boxplus p. & \text{Hence:} \\
\Diamond\boxplus p \xrightarrow{(R\boxplus)} \boxplus\Diamond\boxplus p & \xrightarrow{(a)} \boxplus\Box\Diamond p. & \text{(b)} \\
\Diamond\boxplus p \xrightarrow{(A\boxplus)} \boxplus\Diamond p, & \text{obtained from } (a) \text{ and } (b). &
\end{array}$$

(1') We obtain $\diamondplus\Box p \longrightarrow \Box\diamondplus p$ by duality from (1).

(2) Derive $\diamondplus\boxplus p \longrightarrow \boxplus\diamondplus p$ using (1') similarly (replace $\Diamond$ with $\diamondplus$ above). $\Box$

Note that the two stages of the derivation in the above lemma correspond to two inductions needed to prove that R^+ is convergent, assuming that R is convergent. First, by induction on m, one proves:

$$(x\,R^m\,y \text{ and } x\,R\,z) \ \Rightarrow \ \exists t\colon\ (y\,R\,t \text{ and } z\,R^m\,t).$$

Secondly, by induction on n one proves:

$$(x\,R^m\,y \text{ and } x\,R^n\,z) \ \Rightarrow \ \exists t\colon\ (y\,R^n\,t \text{ and } z\,R^m\,t).$$

Now, if $x\,R^+\,y$ and $x\,R^+\,z$, then $x\,R^m\,y$ and $x\,R^n\,z$, for some m, n. Then there is t such that $y\,R^n\,t$ and $z\,R^m\,t$. Hence $y\,R^+\,t$ and $z\,R^+\,t$. So, R^+ is convergent.

This additionally justifies the name 'induction axiom' for the axiom (A3).

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