On the axiomatisation of common knowledge

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Abstract

Standard axiomatisations of the logic of common knowledge contain the greatest fixed-point axiom schema. While such an inductive principle matches our intuitions in the context of temporal logics, it is not immediately obvious in an epistemic context. We propose an axiom schema that we believe to be more intuitive. It says that if it is common knowledge that everybody knows whether φ then it is common knowledge whether φ. Our schema is sound for KT-based common knowledge and moreover complete for S5-based common knowledge. In contrast, it is unsound for logics without the T-axiom. Our axiom schema directly leads to a simple and intuitive axiomatisation of the 'common knowledge whether' operator.

Keywords: Common knowledge, axiomatisation, induction axiom, greatest fixed-point axiom.

1 Introduction

The standard axiomatisations of the logic of common knowledge contain the induction axiom schema, alias greatest fixed-point axiom

$$\texttt{GFP} \quad \mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\varphi \to \mathbf{C}\varphi),$$

where **C** stands for "it is common knowledge that" and **E** stands for "everybody knows that" [15,13,9]. An alternative axiomatisation [10,6] has the induction rule

$$\texttt{RGFP} \quad \text{from } \varphi \to \mathbf{E}(\psi \wedge \varphi), \text{ infer } \varphi \to \mathbf{C}\psi.$$

In the proof theory literature there exist sequent system counterparts of these principles, e.g. in [1,11]. Similar axioms and rules were used to axiomatise common belief [3,17].

Such inductive principles are common in temporal logics, where they mirror induction on the natural numbers. There, the reading is obvious and the intuitive meaning is clear. More generally, we can make sense of such principles

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when interpreted on well-founded orderings. However, at least to the present authors the meaning of the induction axiom schema is less obvious when the modal operator is that of common knowledge, and one might even wonder whether it is a reasonable principle at all. To witness the difficulty to find an intuitive reading to the above principles, consider the reading of `RGFP` that is given in the introductory chapter of the Handbook of Epistemic Logic:

> "If it is the case that φ is 'self-evident', in the sense that if it is true, then everyone knows it, and, in addition, if φ is true, then everyone knows ψ, we can show by induction that if φ is true, then so is $\mathbf{E}^k(\psi \wedge \varphi)$ for all k." [22]

The explanations in the standard texts resort to concepts such as 'φ indicates to every agent that ψ is true' [16], 'φ is evident' [18], 'it is public that φ is true' [24], or 'φ is a common basis implying shared belief in ψ' [8]. With these understandings `RGFP` can be read "if φ is public and indicates ψ to everybody then truth of φ implies that ψ is common knowledge". The formalisation of these supplementary concepts however introduces further complications, see e.g. [5] for a tentative to settle the logic of 'indicates'.

Can the above inductive principles be replaced by principles with more intuitive appeal? We here propose a new axiom schema:

$$\texttt{GFP0} \quad \mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi).$$

Unlike `GFP` and `RGFP`, it can be read straightforwardly: "if it is common knowledge that everybody knows whether φ then it is common knowledge whether φ"; or alternatively: "common knowledge that the status of φ is shared knowledge implies that the status of φ is common knowledge". In the present paper we focus on KT- and S5-based common knowledge. We prove the following results:

(i) `GFP0` is a theorem if the logic of individual knowledge is at least KT;

(ii) `GFP0` is equivalent to `GFP` if the logic of individual knowledge is S5;

(iii) `GFP0` leads to a simple and intuitive axiomatisation of S5-based 'common knowledge whether';

(iv) `GFP0` is specific to knowledge and fails for belief: contrarily to the status of the standard induction principles, its status differs depending on whether the context is epistemic or doxastic.

Most papers in the literature start by introducing the Kripke semantics and then discuss the axiomatisation of its validities. In contrast, the present paper is semantic-free: all proofs are done syntactically via the axioms and inference rules of the respective systems.

For the sake of simplicity we here only consider shared and common knowledge of the set of all agents. Everything however straightforwardly generalises to common knowledge of arbitrary sets of agents.

The paper is organised as follows. In the next two sections we give the background: two axiom systems for individual knowledge and shared knowledge, KT and S5 (Section 2), and the two standard axiom systems for common

CPC	axiomatics of classical propositional calculus
$\texttt{RN}(\mathbf{K}_i)$	from φ, infer $\mathbf{K}_i\varphi$
$\texttt{K}(\mathbf{K}_i)$	$\mathbf{K}_i(\varphi \to \psi) \to (\mathbf{K}_i\varphi \to \mathbf{K}_i\psi)$
$\texttt{T}(\mathbf{K}_i)$	$\mathbf{K}_i\varphi \to \varphi$
$*\texttt{5}(\mathbf{K}_i)$	$\neg\mathbf{K}_i\varphi \to \mathbf{K}_i\neg\mathbf{K}_i\varphi$
$\texttt{Def}(\mathbf{E})$	$\mathbf{E}\varphi \leftrightarrow \bigwedge_{i\in Agt}\mathbf{K}_i\varphi$

Table 1
Axiomatisation of KT (without $*\texttt{5}(\mathbf{K}_i)$) and S5 (including $*\texttt{5}(\mathbf{K}_i)$) individual knowledge and shared knowledge. The axiom that is not part of the KT axiomatics—i.e., axiom $*\texttt{5}(\mathbf{K}_i)$—is starred.

knowledge (Section 3), which we syntactically prove to be equivalent. In Section 4 we prove that the S5-based GFPO axiomatics is equivalent to the standard axiomatics. In Section 5 we axiomatise S5-based 'common knowledge whether'. In Section 6 we discuss how completeness for logics of knowledge that are weaker than S5 could be obtained. In Section 7 we show that our new axiom is unintuitive for logics of belief, understood as logics that do not have the T axiom for individual belief. We conclude in Section 8.

2 Background: individual and shared knowledge

Let *Prop* be a countable set of propositional variables with typical elements $p, q, \ldots$ Let *Agt* be a fixed, finite set of agents with typical elements $i, j, \ldots$ The grammar of formulas is

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{K}_i\varphi \mid \mathbf{E}\varphi \mid \mathbf{C}\varphi,$$

where p ranges over *Prop* and i over *Agt*. The formula $\mathbf{K}_i\varphi$ reads "i knows that φ"; $\mathbf{E}\varphi$ reads "everybody knows that φ", or "it is shared knowledge that φ";[2] finally, $\mathbf{C}\varphi$ reads "it is common knowledge that φ".

A logic of common knowledge is based on a logic of the individual knowledge operators $\mathbf{K}_i$ that is situated between S5 and KT, where the latter is the weakest normal modal logic having the truth axiom $\mathbf{K}_i\varphi \to \varphi$. In this paper we only consider S5 and KT individual knowledge. Table 1 recalls the two axiomatistions as well as the axiom $\texttt{Def}(\mathbf{E})$ defining shared knowledge. In order to distinguish the axioms and theorems of S5 from those of KT we adopt the convention that formulas that are not theorems of logic KT are prefixed by "$*$", such as $*\texttt{5}(\mathbf{K}_i)$ in Table 1.

The operator $\mathbf{E}$ is a normal modal operator: it obeys the modal schema K and the rule of necessitation RN. Moreover it obeys:

$$\texttt{T}(\mathbf{E}) \quad \mathbf{E}\varphi \to \varphi.$$

[2] Many authors use the adjective 'mutual' instead of 'shared'. We opted for the latter because some philosophers such as Stephen Schiffer use the terms 'mutual knowledge' and 'mutual belief' [20] in order to refer to common knowledge and common belief (see e.g. [14]).

It is straightforward to prove that the following holds for logics of individual knowledge from KT on:

Proposition 2.1 *The formula*

$$\texttt{Def2}(\mathbf{Eif}) \quad (\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \leftrightarrow \bigwedge_{i\in Agt}(\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi)$$

is a theorem of the KT axiomatics.

The name of the equivalence anticipates its use in the axiomatics of 'knowing-whether' in Section 5.

Despite the fact that the shared knowledge operator **E** neither obeys positive nor negative introspection, it obeys the B axiom:

Proposition 2.2 *The formula*

$$*\texttt{B}(\mathbf{E}) \quad \neg\varphi \to \mathbf{E}\neg\mathbf{E}\varphi$$

is a theorem of the S5 axiomatics.

Proof. The proof is simple, but we give it here as we did not find it in the literature:

(i)	$\varphi \to \mathbf{K}_i\neg\mathbf{K}_i\neg\varphi$	$*\texttt{B}(\mathbf{K}_i)$
(ii)	$\neg\mathbf{K}_i\neg\varphi \to \neg\mathbf{E}\neg\varphi$	from $\texttt{Def}(\mathbf{E})$
(iii)	$\mathbf{K}_i\neg\mathbf{K}_i\neg\varphi \to \mathbf{K}_i\neg\mathbf{E}\neg\varphi$	from (ii), $\mathbf{K}_i$ normal
(iv)	$\varphi \to \mathbf{K}_i\neg\mathbf{E}\neg\varphi$	from (i), (iii)
(v)	$\varphi \to \mathbf{E}\neg\mathbf{E}\neg\varphi$	from (iv) with $\texttt{Def}(\mathbf{E})$

□

3 Background: two standard axiomatisations of common knowledge

An overview of the different axiomatisations of logics of common knowledge can be found in [17] where the relation between the underlying logic of individual knowledge and the resulting logic of common knowledge is studied in depth. The paper not only considers knowledge, but also belief. As already said above, our new axiom is not appropriate for common belief. Moreover, only two logics of knowledge are in focus in the present section: systems where the logic of $\mathbf{K}_i$ is either KT or S5. (Logics of knowledge between these two are discussed in Section 6.)

In the next two subsections we recall two standard axiomatisations of the logic of common knowledge, one with the induction rule RGFP and one with the induction axiom schema GFP. We then prove the equivalence of these two axiomatisations.

3.1 With the induction axiom GFP

The two axiomatics with the induction axiom schema GFP are in Table 2 (left). We distinguish the S5-based from the KT-based axiomatics by starring the supplementary axioms, namely the negative introspection axioms $*\texttt{5}(\mathbf{K}_i)$ and

$*5(\mathbf{C})$. Both axiomatics are due to [9]; others can be found in [15,13]. Such axiomatisations are popular in Dynamic Epistemic Logics [21,23].

It is a standard result in normal modal logics that axiom 4 can be proved from T and 5. In the case of common knowledge, $4(\mathbf{C})$ is already a theorem of the KT-based logic thanks to the induction axiom schema:

Proposition 3.1 *The formula*

$$4(\mathbf{C}) \quad \mathbf{C}\varphi \to \mathbf{CC}\varphi$$

is a theorem of the KT-based GFP *axiomatics.*

Proof.

(i)	$\mathbf{C}(\mathbf{C}\varphi \to \mathbf{EC}\varphi)$	from $\mathtt{FP}'$ and $\mathtt{RN}(\mathbf{C})$
(ii)	$\mathbf{C}(\mathbf{C}\varphi \to \mathbf{EC}\varphi) \to (\mathbf{C}\varphi \to \mathbf{CC}\varphi)$	GFP
(iii)	$\mathbf{C}\varphi \to \mathbf{CC}\varphi$	from (i) and (ii)

□

Proposition 3.2 *Axiom* $*5(\mathbf{C})$ *is redundant in the S5-based* GFP *axiomatics.*

Proof.

(i)	$\neg\mathbf{C}\varphi \to \mathbf{K}_i\neg\mathbf{K}_i\mathbf{C}\varphi$	$*\mathtt{B}(\mathbf{K}_i)$
(ii)	$\mathbf{C}\varphi \to \mathbf{K}_i\mathbf{C}\varphi$	from $\mathtt{FP}'$ and $\mathtt{Def}(\mathbf{E})$
(iii)	$\mathbf{K}_i\neg\mathbf{K}_i\mathbf{C}\varphi \to \mathbf{K}_i\neg\mathbf{C}\varphi$	from (ii), $\mathbf{K}_i$ normal
(iv)	$\neg\mathbf{C}\varphi \to \mathbf{K}_i\neg\mathbf{C}\varphi$	from (i), (iii)
(v)	$\neg\mathbf{C}\varphi \to \mathbf{E}\neg\mathbf{C}\varphi$	from (iv) by $\mathtt{Def}(\mathbf{E})$
(vi)	$\mathbf{C}(\neg\mathbf{C}\varphi \to \mathbf{E}\neg\mathbf{C}\varphi)$	from (v) by $\mathtt{RN}(\mathbf{C})$
(vii)	$\mathbf{C}(\neg\mathbf{C}\varphi \to \mathbf{E}\neg\mathbf{C}\varphi) \to (\neg\mathbf{C}\varphi \to \mathbf{C}\neg\mathbf{C}\varphi)$	GFP
(viii)	$\neg\mathbf{C}\varphi \to \mathbf{C}\neg\mathbf{C}\varphi$	from (vi) and (vii)

□

3.2 With the induction rule RGFP

The two axiomatics with the induction rule RGFP are given in Table 2 (right). They are due to [10,6]; the induction rule can actually be traced back to the analysis of common knowledge in the philosophical literature [24]. Interestingly and contrasting with the GFP axiomatics, the S5 axioms and rules for $\mathbf{C}$ are implicit here:

Proposition 3.3 *The formulas* $\mathtt{K}(\mathbf{C})$, $\mathtt{T}(\mathbf{C})$, $4(\mathbf{C})$, *and* $*5(\mathbf{C})$ *are theorems and the rule* $\mathtt{RN}(\mathbf{C})$ *is derivable in the S5-based* RGFP *axiomatics.*

Proof. The proofs are simple, but we give them here for completeness. $\mathtt{K}(\mathbf{C})$ can be proved by substituting φ by $\mathbf{C}\varphi \wedge \mathbf{C}(\varphi \to \psi)$ in RGFP, using FP and that $\mathbf{E}$ is a normal modal operator. $\mathtt{T}(\mathbf{C})$ can be proved from FP and $\mathtt{T}(\mathbf{E})$. $4(\mathbf{C})$ can be proved by substituting both φ and ψ by $\mathbf{C}\varphi$ in RGFP, using FP and that $\mathbf{E}$ is a normal modal operator. The rule $\mathtt{RN}(\mathbf{C})$ can be derived with RGFP if we

GFP-based axiomatics		RGFP-based axiomatics	
KT	the axiomatics of Table 1	KT	the axiomatics of Table 1
RN($\mathbf{C}$)	from φ, infer $\mathbf{C}\varphi$		
K($\mathbf{C}$)	$\mathbf{C}(\varphi \to \psi) \to (\mathbf{C}\varphi \to \mathbf{C}\psi)$		
T($\mathbf{C}$)	$\mathbf{C}\varphi \to \varphi$		
∗5($\mathbf{C}$)	$\neg\mathbf{C}\varphi \to \mathbf{C}\neg\mathbf{C}\varphi$		
FP′	$\mathbf{C}\varphi \to \mathbf{E}\mathbf{C}\varphi$	FP	$\mathbf{C}\varphi \to \mathbf{E}(\varphi \wedge \mathbf{C}\varphi)$
GFP	$\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\varphi \to \mathbf{C}\varphi)$	RGFP	from $\varphi \to \mathbf{E}(\psi \wedge \varphi)$, infer $\varphi \to \mathbf{C}\psi$

Table 2
Two axiomatisations of KT-based and S5-based common knowledge: the GFP axiomatics with an induction axiom of [9] (left) and the RGFP axiomatics with an induction rule of [10,6] (right). The principles that are not part of the KT-based axiomatics—i.e., the ∗5 axioms—are starred.

substitute $\top$ for φ and φ for ψ and use that $\mathbf{E}$ is a normal modal operator. It is only the proof of ∗5($\mathbf{C}$) which is a bit longer:

(i) $\varphi \to \mathbf{E}\neg\mathbf{E}\neg\varphi$ — ∗B($\mathbf{E}$)

(ii) $\mathbf{E}\neg\mathbf{E}\neg\varphi \to \mathbf{E}\neg\mathbf{C}\neg\varphi$ — from FP, $\mathbf{E}$ normal

(iii) $\mathbf{E}\neg\mathbf{C}\neg\varphi \to \mathbf{E}\mathbf{E}\neg\mathbf{E}\mathbf{C}\neg\varphi$ — from ∗B($\mathbf{E}$), $\mathbf{E}$ normal

(iv) $\mathbf{E}\mathbf{E}\neg\mathbf{E}\mathbf{C}\neg\varphi \to \mathbf{E}\mathbf{E}\neg\mathbf{C}\neg\varphi$ — from FP, $\mathbf{E}$ normal

(v) $\mathbf{E}\neg\mathbf{C}\neg\varphi \to \mathbf{E}(\neg\mathbf{C}\neg\varphi \wedge \mathbf{E}\neg\mathbf{C}\neg\varphi)$ — from (iii), (iv), $\mathbf{E}$ normal

(vi) $\mathbf{E}\neg\mathbf{C}\neg\varphi \to \mathbf{C}\neg\mathbf{C}\neg\varphi$ — from (v) by RGFP

(vii) $\varphi \to \mathbf{C}\neg\mathbf{C}\neg\varphi$ — from (i), (ii), (vi)

□

3.3 Equivalence of the two axiomatics

The RGFP axiomatics and the GFP axiomatics are both complete for the same semantics (which we do not give here). Therefore all axioms in one system must be derivable in the other, and the inference rules of one system are admissible in the other. We are however not aware of a direct equivalence proof in the respective systems in the literature, so we give it below. [3] We prove the two directions:

(i) in the RGFP axiomatics, K($\mathbf{C}$), T($\mathbf{C}$), ∗5($\mathbf{C}$), FP′, GFP are theorems and RN($\mathbf{C}$) is derivable;

(ii) in the GFP axiomatics, FP′ is a theorem and RGFP is derivable.

We have already established in Section 3.2 that K($\mathbf{C}$), T($\mathbf{C}$), and ∗5($\mathbf{C}$) are

[3] The paper by Bucheli et al. [4] establishes that RGFP is derivable from a variant of GFP, $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\mathbf{E}\varphi \to \mathbf{C}\varphi)$ (which they have to choose instead of RGFP because they take K as the logic of individual knowledge). However their proof is indirect, making use of an intermediate system.

	the axiomatics of Table 1
RN($\mathbf{C}$)	from φ, infer $\mathbf{C}\varphi$
K($\mathbf{C}$)	$\mathbf{C}(\varphi \to \psi) \to (\mathbf{C}\varphi \to \mathbf{C}\psi)$
4($\mathbf{C}$)	$\mathbf{C}\varphi \to \mathbf{CC}\varphi$
FP0	$\mathbf{C}\varphi \to \mathbf{E}\varphi$
GFP0	$\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$

Table 3
Alternative axiomatisation of S5 common knowledge: the GFP0 axiomatics.

theorems of the RGFP axiomatics. Second and quite obviously, as $\mathbf{E}$ is a normal modal operator, we have that FP′ is provable from FP and that, the other way round, FP is provable from FP′ and T($\mathbf{C}$). It remains to prove the equivalence of the induction axiom and the induction rule.

Proposition 3.4 *The induction axiom* GFP *is a theorem of the KT-based* RGFP *axiomatics (and a fortiori of the S5-based* RGFP *axiomatics).*

Proof.

(i)	$\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{EC}(\varphi \to \mathbf{E}\varphi)$	from FP, $\mathbf{E}$ normal
(ii)	$(\mathbf{C}(\varphi \to \mathbf{E}\varphi) \wedge \varphi) \to \mathbf{E}\varphi$	from T($\mathbf{C}$)
(iii)	$(\mathbf{C}(\varphi \to \mathbf{E}\varphi) \wedge \varphi) \to (\mathbf{E}\varphi \wedge \mathbf{EC}(\varphi \to \mathbf{E}\varphi))$	from (i) and (ii)
(iv)	$(\mathbf{C}(\varphi \to \mathbf{E}\varphi) \wedge \varphi) \to \mathbf{E}((\varphi \wedge \mathbf{C}(\varphi \to \mathbf{E}\varphi)) \wedge \varphi)$	from (iii), $\mathbf{E}$ normal
(v)	$(\mathbf{C}(\varphi \to \mathbf{E}\varphi) \wedge \varphi) \to \mathbf{C}\varphi$	from (iv) by RGFP

□

Proposition 3.5 *The induction rule* RGFP *is derivable in the* GFP *axiomatics.*

Proof.

(i)	$\varphi \to \mathbf{E}(\psi \wedge \varphi)$	hypothesis
(ii)	$\mathbf{C}(\psi \wedge \varphi \to \mathbf{E}(\psi \wedge \varphi))$	from (i) by RN($\mathbf{C}$)
(iii)	$\mathbf{C}(\psi \wedge \varphi \to \mathbf{E}(\psi \wedge \varphi)) \to (\psi \wedge \varphi \to \mathbf{C}(\psi \wedge \varphi))$	GFP
(iv)	$\psi \wedge \varphi \to \mathbf{C}(\psi \wedge \varphi)$	from (ii), (iii)
(v)	$\varphi \to \psi \wedge \varphi$	from (i) by T($\mathbf{E}$)
(vi)	$\varphi \to \mathbf{C}\psi$	from (v), (iv), $\mathbf{C}$ normal

□

4 An alternative axiomatisation of S5 common knowledge

Table 3 contains a new axiomatics of common knowledge. The main difference w.r.t. the GFP axiomatics is that the induction axiom GFP is replaced by GFP0. A further difference is that our axiomatics explicits 4($\mathbf{C}$), which is a theorem of the GFP and RGFP axiomatics. Finally and thanks to 4($\mathbf{C}$), our version of the fixed-point axiom FP0 is weaker than FP′ (and a fortiori weaker than FP). It is however strong enough to entail $\texttt{T}(\mathbf{C})$: $\mathbf{C}\varphi \to \varphi$ (together with $\texttt{Def}(\mathbf{E})$ and $\texttt{T}(\mathbf{K}_i)$).

Observe that it follows from Proposition 2.1 and the fact that $\mathbf{C}$ is a normal modal operator that the two axioms

$$\texttt{GFP0} \quad \mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$$
$$\texttt{GFP1} \quad \mathbf{C}\bigwedge_{i\in Agt}(\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$$

are equivalent. The second axiom says that if it is common knowledge that each agent has an epistemic position w.r.t. φ then either φ or $\neg\varphi$ are common knowledge.

4.1 Soundness of the GFP0 axiomatics

We prove soundness w.r.t. the S5-based GFP axiomatics of Table 2. The result holds both for the KT-based and the S5-based version.

The inference rules are the same: $\texttt{RN}(\mathbf{C})$ and modus ponens. It remains to show that our axioms of Table 3 are theorems of the S5-based GFP axiomatics. The only ones that are missing there are $\texttt{4}(\mathbf{C})$, FP0, and GFP0. First, $\texttt{4}(\mathbf{C})$ is, by Proposition 3.1, a theorem of the KT-based GFP axiomatics and a fortiori of the S5-based GFP axiomatics. Second, FP0 can be proved from FP′ and $\texttt{T}(\mathbf{C})$. Third, here is a proof of GFP0 that relies on $\texttt{T}(\mathbf{K}_i)$, or rather, its consequence $\texttt{T}(\mathbf{E})$:

Proposition 4.1 GFP0 *is a theorem of the KT-based* GFP *axiomatics (and a fortiori of the S5-based* GFP *axiomatics).*

Proof. We distinguish the two cases φ and $\neg\varphi$ and prove that $\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi)$ implies both $\varphi \to \mathbf{C}\varphi$ and $\neg\varphi \to \mathbf{C}\neg\varphi$; from that GFP0 follows by propositional logic reasoning.

(i) $\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to \mathbf{C}(\varphi \to \mathbf{E}\varphi)$ by $\texttt{T}(\mathbf{E})$, $\texttt{RN}(\mathbf{C})$, $\texttt{K}(\mathbf{C})$

(ii) $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\varphi \to \mathbf{C}\varphi)$ GFP

(iii) $\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\varphi \to \mathbf{C}\varphi)$ from (i), (ii)

(iv) $\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\neg\varphi \to \mathbf{C}\neg\varphi)$ from (iii) by uniform subst. of φ by $\neg\varphi$

(v) $\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$ from (iii), (iv)

□

Therefore all theorems of our new GFP0 axiomatics are also theorems of the GFP axiomatics and, by Proposition 3.5, of the RGFP axiomatics.

4.2 Completeness of the GFP0 axiomatics for S5 knowledge

We prove completeness w.r.t. the S5-based GFP axiomatics. We have already seen in Section 4.1 that the inference rules are the same; it remains to show that the axioms of the S5-based GFP axiomatics of Table 2 that are not in our GFP0 axiomatics are theorems of the latter. These axioms are $*\texttt{5}(\mathbf{C})$, FP′, and GFP. Proposition 3.2 tells us that $*\texttt{5}(\mathbf{C})$ can be proved from the rest of the S5-based GFP axiomatics and is therefore redundant: it could be dropped from the GFP axiomatics. Axiom FP′ can be proved from our FP0, $\texttt{4}(\mathbf{C})$, $\texttt{K}(\mathbf{C})$, and

$\texttt{RN}(\mathbf{C})$. It remains to show that $\texttt{GFP}$ is a theorem of our new axiomatics. The next lemma will be instrumental; its proof uses $*\texttt{B}(\mathbf{E})$ (via Proposition 2.2) and $\texttt{4}(\mathbf{C})$.

Lemma 4.2 *The schema* $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$ *is provable from the axiom schemas* $\texttt{K}(\mathbf{C})$, $\texttt{4}(\mathbf{C})$, $\texttt{RN}(\mathbf{C})$, $\texttt{FP}$, $\texttt{Def}(\mathbf{E})$, *and the S5 axioms for* $\mathbf{K}_i$.

Proof. The proof is as follows:

(i)	$\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{E}(\varphi \to \mathbf{E}\varphi)$	by $\texttt{FP}$, $\mathbf{E}$ normal
(ii)	$\mathbf{E}(\varphi \to \mathbf{E}\varphi) \to (\mathbf{E}\neg\mathbf{E}\varphi \to \mathbf{E}\neg\varphi)$	$\mathbf{E}$ normal
(iii)	$\neg\varphi \to \mathbf{E}\neg\mathbf{E}\varphi$	Proposition 2.2
(iv)	$\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\neg\varphi \to \mathbf{E}\neg\varphi)$	from (i), (ii), (iii)
(v)	$\mathbf{CC}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$	from (iv) by $\texttt{RN}(\mathbf{C})$ and $\texttt{K}(\mathbf{C})$
(vi)	$\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{CC}(\varphi \to \mathbf{E}\varphi)$	$\texttt{4}(\mathbf{C})$
(vii)	$\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$	from (v), (vi)

□

Proposition 4.3 $\texttt{GFP}$ *is provable in the* $\texttt{GFP0}$ *axiomatics.*

Proof. The proof is as follows:

(i)	$\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$	$\texttt{GFP0}$
(ii)	$\big(\mathbf{C}(\varphi \to \mathbf{E}\varphi) \wedge \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)\big) \to \mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi)$	by $\texttt{RN}(\mathbf{C})$ and $\texttt{K}(\mathbf{C})$
(iii)	$\big(\mathbf{C}(\varphi \to \mathbf{E}\varphi) \wedge \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)\big) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$	from (i) and (ii)
(iv)	$\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$	Lemma 4.2
(v)	$\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$	from (iii), (iv)
(vi)	$\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\mathbf{C}\varphi \vee \neg\varphi)$	from (v) by $\texttt{T}(\mathbf{C})$

□

5 Commonly knowing whether

In this section we show that our axiomatics of Table 3 leads to a simple axiomatisation of the S5-based 'common knowledge whether' operator. The axiomatisation of such an operator was left as an open problem in [7], where operators of 'individually knowing whether' were axiomatised.

The first thing we do is to extend our language of "knowing-that" operators $\mathbf{K}_i$, $\mathbf{E}$, and $\mathbf{C}$ by their "knowing-whether" counterparts. We read $\mathbf{Kif}_i\varphi$ as "i knows whether φ"; $\mathbf{Eif}\varphi$ as "it is shared knowledge whether φ"; and $\mathbf{Cif}\varphi$ as "it is common knowledge whether φ". These three epistemic operators are particular modal operators of contingency [19,12,7].

A straightforward possibility is to add to the axiomatics of Table 3 the following axioms:

$\texttt{Def1}(\mathbf{Kif}_i)$	$\mathbf{Kif}_i\varphi \leftrightarrow (\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi)$
$\texttt{Def1}(\mathbf{Eif})$	$\mathbf{Eif}\varphi \leftrightarrow (\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi)$
$\texttt{Def1}(\mathbf{Cif})$	$\mathbf{Cif}\varphi \leftrightarrow (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$

CPC	axiomatics of classical propositional calculus
Sym($\mathbf{Kif}_i$)	$\mathbf{Kif}_i\varphi \leftrightarrow \mathbf{Kif}_i\neg\varphi$
RE($\mathbf{Kif}_i$)	from $\varphi \leftrightarrow \psi$, infer $\mathbf{Kif}_i\varphi \leftrightarrow \mathbf{Kif}_i\psi$
RN($\mathbf{Kif}_i$)	from φ, infer $\mathbf{Kif}_i\varphi$
Conj($\mathbf{Kif}_i$)	$(\varphi \wedge \psi) \to \big(\mathbf{Kif}_i(\varphi \wedge \psi) \leftrightarrow (\mathbf{Kif}_i\varphi \wedge \mathbf{Kif}_i\psi)\big)$
$*45_1$($\mathbf{Kif}_i$)	$\mathbf{Kif}_i\mathbf{Kif}_i\varphi$
$*45_2$($\mathbf{Kif}_i$)	$\mathbf{Kif}_i(\varphi \wedge \mathbf{Kif}_i\varphi)$
Def2($\mathbf{Eif}$)	$\mathbf{Eif}\varphi \leftrightarrow \bigwedge_{i\in Agt}\mathbf{Kif}_i\varphi$
Sym($\mathbf{Cif}$)	$\mathbf{Cif}\varphi \leftrightarrow \mathbf{Cif}\neg\varphi$
RE($\mathbf{Cif}$)	from $\varphi \leftrightarrow \psi$, infer $\mathbf{Cif}\varphi \leftrightarrow \mathbf{Cif}\psi$
RN($\mathbf{Cif}$)	from φ, infer $\mathbf{C}\varphi$
Conj($\mathbf{Cif}$)	$(\varphi \wedge \psi) \to \big(\mathbf{Cif}(\varphi \wedge \psi) \leftrightarrow (\mathbf{Cif}\varphi \wedge \mathbf{Cif}\psi)\big)$
$*45_1$($\mathbf{Cif}$)	$\mathbf{Cif}\mathbf{Cif}\varphi$
$*45_2$($\mathbf{Cif}$)	$\mathbf{Cif}(\varphi \wedge \mathbf{Cif}\varphi)$
GFP2	$\mathbf{Cif}\varphi \leftrightarrow (\mathbf{Eif}\varphi \wedge \mathbf{Cif}\mathbf{Eif}\varphi)$
Def2($\mathbf{K}_i$)	$\mathbf{K}_i\varphi \leftrightarrow (\varphi \wedge \mathbf{Kif}_i\varphi)$
Def2($\mathbf{E}$)	$\mathbf{E}\varphi \leftrightarrow (\varphi \wedge \mathbf{Eif}\varphi)$
Def2($\mathbf{C}$)	$\mathbf{C}\varphi \leftrightarrow (\varphi \wedge \mathbf{Cif}\varphi)$

Table 4
Axiomatisation of S5 common knowledge whether: the GFP2 axiomatics.

However, we are going to take another road here, in view of axiomatising the fragment without 'knowing-that' operators. Our axiomatics in Table 4 takes the 'knowing-whether' operators as primitive and defines the 'knowing-that' operators. The first part is proper to $\mathbf{Kif}_i$ and $\mathbf{Eif}$. We might have taken over as well the axiomatics of [7]; the principles Sym($\mathbf{Kif}_i$), RE($\mathbf{Kif}_i$), and RN($\mathbf{Kif}_i$) can also be found there, but we find the rest of our axioms a bit simpler than theirs. Axiom 45_1($\mathbf{Kif}_i$) can be found in [19]. The second part of our axiomatics parallels the first part and moreover has a single greatest fixed-point axiom relating $\mathbf{Eif}$ and $\mathbf{Cif}$ (that is perhaps better called a fixed-point axiom *tout court*: its syntactical form is very close to that of a possible fixed-point axiom for common belief $\mathbf{CB}\,\varphi \leftrightarrow (\mathbf{EB}\,\varphi \wedge \mathbf{EB}\ \mathbf{CB}\,\varphi)$). The third part contains the definitions of the 'knowing-that' operators.

We are going to prove soundness and completeness of the axiomatics of Table 4 w.r.t. the S5-based GFP0 axiomatics (more precisely: w.r.t. the extension of the latter by Def1($\mathbf{Kif}_i$), Def1($\mathbf{Eif}$), and Def1($\mathbf{Cif}$)).

Proposition 5.1 *For the S5-based* GFP2 *axiomatics of Table 4, all inference rules are derivable and all axioms are theorems in the S5-based* GFP0 *axiomatics.*

Proof. See the appendix. □

Proposition 5.2 *For the S5-based* GFP0 *axiomatics of Table 3, all inference rules are derivable and all axioms are theorems in the S5-based* GFP2 *axiomatics. Moreover, the equivalences* Def1($\mathbf{K}_i$)*,* Def1($\mathbf{E}$)*, and* Def1($\mathbf{C}$) *are theorems in the S5-based* GFP2 *axiomatics.*

Proof. See the appendix. □

It follows from propositions 5.1 and 5.2 that the first two parts of Table 4 provide a sound and complete axiomatisation for the fragment of the language with only 'knowing-whether' operators.

Proposition 5.3 *If formula φ has no $\mathbf{K}_i$, $\mathbf{E}$, $\mathbf{C}$ operators then φ is a theorem of the S5-based* `GFP2` *axiomatics of Table 4 if and only if it is provable without the axioms* $\mathtt{Def2}(\mathbf{K}_i)$*,* $\mathtt{Def2}(\mathbf{E})$*, and* $\mathtt{Def2}(\mathbf{C})$*.*

Proof. Suppose no $\mathbf{K}_i$, $\mathbf{E}$, $\mathbf{C}$ occur in φ and suppose φ is a theorem of the S5-based `GFP2` axiomatics. Whenever the proof of φ uses axiom $\mathtt{Def2}(\mathbf{K}_i)$, $\mathtt{Def2}(\mathbf{E})$, or $\mathtt{Def2}(\mathbf{C})$, we can eliminate that axiom by replacing the *definiendum* by the *definiens* everywhere in the proof. □

We end this section by a comment on alternative definitions of 'knowing-whether' group attitudes. As noted in the conclusion of [7], there are more options than those we have considered in this section. We have chosen to define 'shared knowledge whether' as $\mathbf{Eif}\varphi \leftrightarrow (\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi)$. However, instead of requiring that everybody has the same epistemic position about φ one could only require that everybody has *some* epistemic position about φ. This amounts to defining 'weak shared knowledge whether' by $\mathbf{Eif}^w\varphi \leftrightarrow \bigwedge_{i\in Agt}\mathbf{Kif}_i\varphi$. At first glance this is a less demanding notion; however, Proposition 2.1 tells us that $\mathbf{Eif}$ and $\mathbf{Eif}^w$ are equivalent as soon as KT is our basic epistemic logic. Similarly, seemingly weaker definitions of 'common knowledge whether' exist. Instead of requiring that either φ or $\neg\varphi$ is common knowledge, one could only require (a) that it is common knowledge that there is shared knowledge whether φ, or (b) that it is common knowledge that there is weak shared knowledge whether φ. This amounts to replacing $\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi$ in the definition of 'common knowledge whether' either by $\mathbf{C}\,\mathbf{Eif}\varphi$, or by $\mathbf{C}\,\mathbf{Eif}^w\varphi$. Again, these two definitions appear to be weaker than ours, but this fails to be the case. This can be seen from the theorem

$$\mathtt{GFP1} \quad \mathbf{C}\bigwedge_{i\in Agt}(\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$$

of Section 4 (end of the second paragraph) and that can be reformulated as $\mathbf{CEif}^w\varphi \to \mathbf{Cif}\varphi$. We note that both for shared and common knowledge whether, the two options are no longer equivalent for weaker logics, i.e., for logics of belief. We will come back to this in Section 7.

6 Discussion: epistemic logics between KT and S5

We have seen that our new axiom `GFP0` is sound for logics of knowledge, understood as logics where the logic of individual knowledge is at least KT, and that it is complete when the logic of individual knowledge is S5.

We conjecture that the KT-based `GFP0` axiomatics is incomplete. We however do not have a formal proof for the time being. Such a proof would have to delve into semantics: it typically consists in designing a non-standard semantics for which the axiomatics with `GFP0` is complete. We leave this aside for the time being.

Under the hypothesis that the KT-based `GFP0` axiomatics is incomplete, one may wonder which axiom is missing to obtain completeness. A tempting avenue is to add the formula $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$ of Lemma 4.2 as an axiom schema to the axiomatics of Table 3. The proof of Proposition 4.3 then gives us completeness because it uses none of the S5 axioms but $\mathtt{T}(\mathbf{K}_i)$. However it can be shown that this amounts to adding $*\mathtt{5}(\mathbf{C})$: it can be shown that the formula is equivalent to $*\mathtt{5}(\mathbf{C})$ in the presence of $\mathtt{T}(\mathbf{C})$.

Proposition 6.1 *In the GFP-based axiomatics for KT,* $*\mathtt{5}(\mathbf{C})$ *and the formula* $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$ *are interderivable.*

Proof. See the appendix. □

Just as common knowledge is necessarily positively introspective even when individual knowledge isn't, it can still be argued that S5 common knowledge can make sense even when individual knowledge is not S5: one can imagine, e.g., that common knowledge is "written on a blackboard", or otherwise easily available to agents such that they are able to immediately verify what is and is not commonly known. We leave further explorations to future work.

7 Discussion: `GFP0` is not appropriate for belief

Up to now we have only discussed common knowledge; we now briefly discuss common belief.

Let us write $\mathbf{B}_i\,\varphi$ for "i believes that φ", $\mathbf{EB}\,\varphi$ for "it is shared belief that φ", and $\mathbf{CB}\,\varphi$ for "it is common belief that φ", and let us suppose the logic of the $\mathbf{B}_i$ operators is KD (or, alternatively, any logic without the `T` axiom).

It is intuitively clear that the belief-version of `GFP1`,

$$\mathbf{CB} \bigwedge_{i \in Agt} (\mathbf{B}_i\,\varphi \lor \mathbf{B}_i\,\neg\varphi) \to (\mathbf{CB}\,\varphi \lor \mathbf{CB}\,\neg\varphi),$$

should not hold: if there is common belief—and even common knowledge—that everybody has an opinion about φ then it by no means follows that there is common belief about φ.

What about `GFP0`? The fact that `GFP1` is unintuitive need not disqualify `GFP0`. Indeed, while these two axioms are equivalent in epistemic contexts, they fail to be so in doxastic contexts: in KD45, $\bigwedge_{i \in Agt}(\mathbf{B}_i\,\varphi \lor \mathbf{B}_i\,\neg\varphi)$ does not imply $\mathbf{EB}\,\varphi \lor \mathbf{EB}\,\neg\varphi$, and does not do so a fortiori in KD; and therefore the belief-counterpart of Proposition 2.1 does not hold.

As it turns out, `GFP0` is not a reasonable principle of common belief either. This can be highlighted by the following example. Suppose that the set of agents under concern is $Agt = \{1, 2\}$ and that there is a misunderstanding between 1 and 2 about an inform act of a third agent. That third agent is not relevant here, and we suppose that $Agt = \{1, 2\}$. Let us suppose that 1 believes the third agent said p and therefore believes that p is in the common ground ($\mathbf{B}_1\,\mathbf{CB}\,p$), while 2 believes that $\neg p$ is in the common ground ($\mathbf{B}_2\,\mathbf{CB}\,\neg p$). It follows by $\mathtt{4}(\mathbf{CB})$ and by the (intuitively still valid) belief-counterpart of `FP0` that

$$\mathbf{B}_1\,\mathbf{CB}\,\mathbf{EB}\,p \land \mathbf{B}_2\,\mathbf{CB}\,\mathbf{EB}\,\neg p.$$

As both $\mathbf{CB}$ and $\mathbf{EB}$ are normal operators, it follows that

$$\mathbf{B}_1\,\mathbf{CB}\,(\mathbf{EB}\,p \lor \mathbf{EB}\,\neg p) \land \mathbf{B}_2\,\mathbf{CB}\,(\mathbf{EB}\,p \lor \mathbf{EB}\,\neg p),$$

i.e., that $\mathbf{EB}\,\mathbf{CB}\,(\mathbf{EB}\,p \lor \mathbf{EB}\,\neg p)$. The latter is equivalent to $\mathbf{CB}\,(\mathbf{EB}\,p \lor \mathbf{EB}\,\neg p)$ thanks to the belief-version of the fixed-point axiom, which is $\mathbf{CB}\,\varphi \leftrightarrow \mathbf{EB}\,\mathbf{CB}\,\varphi$.

From that the counter-intuitive consequence $\mathbf{CB}\,p \lor \mathbf{CB}\,\neg p$ would follow by the belief-counterpart of GFP0.

To sum up, contrarily to the status of the standard induction principles the status of our new versions of the induction axiom differs between knowledge and belief: they are specific to common knowledge and fail for common belef.

8 Conclusion

We have studied the axiomatisation of the logic of common knowledge, coming up with an alternative GFP0 to the standard induction axiom principles that is intuitively appealing as an axiom for common knowledge. While our proofs are not very difficult, we believe that GFP0 will lead to presentations of epistemic logic that are intuitively more appealing.

Our investigation may appear somewhat old-fashioned: all our proofs are purely syntactical and we do not use any semantical tools, as was done in ‘the syntactic era (1918-1959)’ [2, Section 1.7] before Kripke semantics was invented. We nevertheless believe that axiomatic systems provide an important toolbox to understand intuitively what a logical system is able to express and what not. To witness, consider the inference rule RGFP: according to the explanations e.g. in [24], the rule says something about φ indicating to everybody that ψ; however and as the equivalence with axiom GFP demonstrates, this is not the case: axiom GFP of the equivalent GFP-based axiomatics has a single schematic variable φ, which shows us that the concept of one proposition indicating another proposition is not accounted for by the Kripke semantics. This is in line with the analysis of [5] where it is argued that this concept cannot be modelled in Kripke semantics and where the authors investigate a different semantical framework.

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Appendix

A Proofs of Section 5

A.1 Proof of Proposition 5.1

Proposition 5.1 *For the S5-based* `GFP2` *axiomatics of Table 4, all inference rules are derivable and all axioms are theorems in the S5-based* `GFP0` *axiomatics.*

We prove each principle of Table 4. We start by the last three definitions so that we can use them in the rest of the proofs.

`Def2`$(\mathbf{K}_i)$ $\quad \mathbf{K}_i\varphi \leftrightarrow (\varphi \wedge \mathbf{Kif}_i\varphi)$

Proof.

(i)	$\mathbf{K}_i\varphi \leftrightarrow (\varphi \wedge (\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi))$	from `T`$(\mathbf{K}_i)$
(ii)	$\mathbf{K}_i\varphi \leftrightarrow (\varphi \wedge \mathbf{Kif}_i\varphi)$	from (i) and `Def1`$(\mathbf{Kif}_i)$

□

`Def2`$(\mathbf{E})$ $\quad \mathbf{E}\varphi \leftrightarrow (\varphi \wedge \mathbf{Eif}\varphi)$

Proof. Follow the lines of that of `Def2`$(\mathbf{K}_i)$: use `Def1`$(\mathbf{Eif})$ instead of `Def1`$(\mathbf{Kif}_i)$ and use that `T`$(\mathbf{E})$ is a theorem. □

`Def2`$(\mathbf{C})$ $\quad \mathbf{C}\varphi \leftrightarrow (\varphi \wedge \mathbf{Cif}\varphi)$

Proof. Follow the lines of that of `Def2`$(\mathbf{K}_i)$: use `Def1`$(\mathbf{Cif})$ instead of `Def1`$(\mathbf{Kif}_i)$ and `T`$(\mathbf{C})$ instead of `T`$(\mathbf{K}_i)$. □

`Sym`$(\mathbf{Kif}_i)$: $\mathbf{Kif}_i\varphi \leftrightarrow \mathbf{Kif}_i\neg\varphi$

Proof.

(i)	$(\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi) \leftrightarrow (\mathbf{K}_i\neg\varphi \vee \mathbf{K}_i\neg\neg\varphi)$	$\mathbf{K}_i$ normal
(ii)	$\mathbf{Kif}_i\varphi \leftrightarrow \mathbf{Kif}_i\neg\varphi$	from (i) by `Def1`$(\mathbf{Kif}_i)$

□

`RE`$(\mathbf{Kif}_i)$: from $\varphi \leftrightarrow \psi$, infer $\mathbf{Kif}_i\varphi \leftrightarrow \mathbf{Kif}_i\psi$

Proof.

(i)	$\varphi \leftrightarrow \psi$	hypothesis
(ii)	$\mathbf{K}_i\varphi \leftrightarrow \mathbf{K}_i\psi$	from (i), $\mathbf{K}_i$ normal
(iii)	$\mathbf{K}_i\neg\varphi \leftrightarrow \mathbf{K}_i\neg\psi$	from (i), $\mathbf{K}_i$ normal
(iv)	$(\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi) \leftrightarrow (\mathbf{K}_i\psi \vee \mathbf{K}_i\neg\psi)$	from (ii), (iii)
(v)	$\mathbf{Kif}_i\varphi \leftrightarrow \mathbf{Kif}_i\psi$	from (iv) by `Def1`$(\mathbf{Kif}_i)$

□

$\texttt{RN}(\mathbf{Kif}_i)$: from φ, infer $\mathbf{Kif}_i\varphi$

Proof.

(i)	φ	hypothesis
(ii)	$\mathbf{K}_i\varphi$	from (i), $\mathbf{K}_i$ normal
(iii)	$\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi$	from (ii)
(iv)	$\mathbf{Kif}_i\varphi$	from (iii) by $\texttt{Def1}(\mathbf{Kif}_i)$

□

$\texttt{Conj}(\mathbf{Kif}_i)$: $(\varphi \wedge \psi) \to \big(\mathbf{Kif}_i(\varphi \wedge \psi) \leftrightarrow (\mathbf{Kif}_i\varphi \wedge \mathbf{Kif}_i\psi)\big)$

Proof. We prove the two implications $\big(\varphi \wedge \psi \wedge \mathbf{Kif}_i(\varphi \wedge \psi)\big) \to \mathbf{Kif}_i\varphi$ and $(\varphi \wedge \psi \wedge \mathbf{Kif}_i\varphi \wedge \mathbf{Kif}_i\psi) \to \mathbf{Kif}_i(\varphi \wedge \psi)$, each time using that we have already proved $\texttt{Def2}(\mathbf{K}_i)$ to be a theorem. For the former:

(i)	$\mathbf{K}_i(\varphi \wedge \psi) \to (\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi)$	$\mathbf{K}_i$ normal
(ii)	$\big(\varphi \wedge \psi \wedge \mathbf{Kif}_i(\varphi \wedge \psi)\big) \to \mathbf{Kif}_i\varphi$	from (i), theorem $\texttt{Def2}(\mathbf{K}_i)$

For the latter:

(i)	$(\mathbf{K}_i\varphi \wedge \mathbf{K}_i\psi) \to \mathbf{K}_i(\varphi \wedge \psi)$	$\mathbf{K}_i$ normal
(ii)	$(\varphi \wedge \mathbf{Kif}_i\varphi \wedge \psi \wedge \mathbf{Kif}_i\psi) \to \big(\varphi \wedge \psi \wedge \mathbf{Kif}_i(\varphi \wedge \psi)\big)$	from (i), thm. $\texttt{Def2}(\mathbf{K}_i)$
(iii)	$(\varphi \wedge \psi \wedge \mathbf{Kif}_i\varphi \wedge \mathbf{Kif}_i\psi) \to \mathbf{Kif}_i(\varphi \wedge \psi)$	from (ii)

□

$\texttt{45}_1(\mathbf{Kif}_i)$: $\mathbf{Kif}_i\mathbf{Kif}_i\varphi$

Proof. Similar to the next proof of $\texttt{45}_2(\mathbf{Kif}_i)$. □

$\texttt{45}_2(\mathbf{Kif}_i)$: $\mathbf{Kif}_i(\varphi \wedge \mathbf{Kif}_i\varphi)$

Proof.

(i)	$\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi \vee (\neg\mathbf{K}_i\varphi \wedge \neg\mathbf{K}_i\neg\varphi)$	
(ii)	$\mathbf{K}_i\varphi \to \mathbf{K}_i(\varphi \wedge \mathbf{Kif}_i\varphi)$	from $\texttt{4}(\mathbf{K}_i)$ and thm. $\texttt{Def2}(\mathbf{K}_i)$, $\mathbf{K}_i$ normal
(iii)	$\mathbf{K}_i\neg\varphi \to \mathbf{K}_i\neg(\varphi \wedge \mathbf{Kif}_i\varphi)$	from $\mathbf{K}_i$ normal
(iv)	$(\neg\mathbf{K}_i\varphi \wedge \neg\mathbf{K}_i\neg\varphi) \to (\mathbf{K}_i\neg\mathbf{K}_i\varphi \wedge \mathbf{K}_i\neg\mathbf{K}_i\neg\varphi)$	from thm. $*\texttt{5}(\mathbf{K}_i)$
(v)	$(\mathbf{K}_i\neg\mathbf{K}_i\varphi \wedge \mathbf{K}_i\neg\mathbf{K}_i\neg\varphi) \to \mathbf{K}_i\neg\mathbf{Kif}_i\varphi$	from $\texttt{Def1}(\mathbf{Kif}_i)$, $\mathbf{K}_i$ normal
(vi)	$(\neg\mathbf{K}_i\varphi \wedge \neg\mathbf{K}_i\neg\varphi) \to \mathbf{K}_i\neg(\varphi \wedge \mathbf{Kif}_i\varphi)$	from (iv), (v), $\mathbf{K}_i$ normal
(vii)	$\mathbf{K}_i(\varphi \wedge \mathbf{Kif}_i\varphi) \vee \mathbf{K}_i\neg(\varphi \wedge \mathbf{Kif}_i\varphi)$	from (i), (ii), (iii), (vi)
(viii)	$\mathbf{Kif}_i(\varphi \wedge \mathbf{Kif}_i\varphi)$	from (vii), $\texttt{Def1}(\mathbf{Kif}_i)$

□

$$\texttt{Def2}(\mathbf{Eif})\text{: } \mathbf{Eif}\varphi \leftrightarrow \bigwedge_{i \in Agt} \mathbf{Kif}_i\varphi$$

Proof. This is Proposition 2.1. □

$$\texttt{Sym}(\mathbf{Cif})\text{: } \mathbf{Cif}\varphi \leftrightarrow \mathbf{Cif}\neg\varphi$$

Proof. Follow the lines of that of $\texttt{Sym}(\mathbf{Kif}_i)$. □

$$\texttt{RE}(\mathbf{Cif})\text{: from } \varphi \leftrightarrow \psi\text{, infer } \mathbf{Cif}\varphi \leftrightarrow \mathbf{Cif}\psi$$

Proof. Follow the lines of that of $\texttt{RE}(\mathbf{Kif}_i)$. □

$$\texttt{RN}(\mathbf{Cif}) \quad \text{from } \varphi\text{, infer } \mathbf{C}\varphi$$

Proof. Follow the lines of that of $\texttt{RN}(\mathbf{Kif}_i)$. □

$$\texttt{Conj}(\mathbf{Cif}) \quad (\varphi \wedge \psi) \to \big(\mathbf{Cif}(\varphi \wedge \psi) \leftrightarrow (\mathbf{Cif}\varphi \wedge \mathbf{Cif}\psi)\big)$$

Proof. Follow the lines of that of $\texttt{Conj}(\mathbf{Kif}_i)$. □

$$*\texttt{45}_1(\mathbf{Cif}) \quad \mathbf{Cif}\mathbf{Cif}\varphi$$

Proof. Follow the lines of that of $\texttt{45}_1(\mathbf{Kif}_i)$. □

$$\texttt{45}_2(\mathbf{Cif}) \quad \mathbf{Cif}(\varphi \wedge \mathbf{Cif}\varphi)$$

Proof. Follow the lines of that of $\texttt{45}_2(\mathbf{Kif}_i)$. □

$$\texttt{GFP2} \quad \mathbf{Cif}\varphi \leftrightarrow (\mathbf{Eif}\varphi \wedge \mathbf{Cif}\mathbf{Eif}\varphi)$$

Proof. We prove the three implications $\mathbf{Cif}\varphi \to \mathbf{Eif}\varphi$, $\mathbf{Cif}\varphi \to \mathbf{Cif}\mathbf{Eif}\varphi$, and $(\mathbf{Eif}\varphi \wedge \mathbf{Cif}\mathbf{Eif}\varphi) \to \mathbf{Cif}\varphi$. For the first:

(i) $(\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi) \to (\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi)$ — from $\texttt{FP0}$

(ii) $\mathbf{Cif}\varphi \to \mathbf{Eif}\varphi$ — from (i), $\texttt{Def1}(\mathbf{Eif})$, $\texttt{Def1}(\mathbf{Cif})$

For the second:

(i) $\mathbf{C}\varphi \to \mathbf{CE}\varphi$ — from $\texttt{4}(\mathbf{C})$, $\texttt{FP0}$

(ii) $\mathbf{C}\varphi \to \mathbf{CEif}\varphi$ — from (i), $\texttt{Def}(\mathbf{Eif})$, normal $\mathbf{C}$

(iii) $\mathbf{C}\neg\varphi \to \mathbf{CEif}\neg\varphi$ — from (ii) by uniform substitution

(iv) $\mathbf{C}\neg\varphi \to \mathbf{CEif}\varphi$ — from (iii) by $\texttt{Sym}(\mathbf{K}_i)$, $\texttt{Def}(\mathbf{E})$

(v) $\mathbf{Cif}\varphi \to \mathbf{CEif}\varphi$ — from (ii), (iv), $\texttt{Def1}(\mathbf{Cif})$

(vi) $\mathbf{Cif}\varphi \to \mathbf{Cif}\mathbf{Eif}\varphi$ — from (v), $\texttt{Def1}(\mathbf{Cif})$

For the third:

(i) $\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$ GFP0

(ii) $\mathbf{CEif}\varphi \to \mathbf{Cif}\varphi$ from (i), Def1($\mathbf{Eif}$), Def1($\mathbf{Cif}$)

(iii) $(\mathbf{Eif}\varphi \wedge \mathbf{CifEif}\varphi) \to \mathbf{Cif}\varphi$ from (ii), thm. Def2($\mathbf{C}$)

□

A.2 Proof of Proposition 5.2

Proposition 5.2 *For the S5-based* GFP0 *axiomatics of Table 3, all inference rules are derivable and all axioms are theorems in the S5-based* GFP2 *axiomatics. Moreover, the equivalences* Def1($\mathbf{Kif}_i$), Def1($\mathbf{Eif}$), *and* Def1($\mathbf{Cif}$) *are theorems in the S5-based* GFP2 *axiomatics.*

We start by the last three definitions.

Def1($\mathbf{Kif}_i$)	$\mathbf{Kif}_i\varphi \leftrightarrow (\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi)$

Proof.

(i) $(\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi) \leftrightarrow ((\varphi \wedge \mathbf{Kif}_i\varphi) \vee (\neg\varphi \wedge \mathbf{Kif}_i\neg\varphi))$ from Def2($\mathbf{K}_i$)

(ii) $\mathbf{Kif}_i\neg\varphi \leftrightarrow \mathbf{Kif}_i\varphi$ Sym($\mathbf{Kif}_i$)

(iii) $\mathbf{Kif}_i\varphi \leftrightarrow (\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi)$ from (i), (ii)

□

Def1($\mathbf{Eif}$)	$\mathbf{Eif}\varphi \leftrightarrow (\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi)$

Proof. Follow the lines of that of Def1($\mathbf{Kif}_i$). □

Def1($\mathbf{Cif}$)	$\mathbf{Cif}\varphi \leftrightarrow (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$

Proof. Follow the lines of that of Def1($\mathbf{Kif}_i$). □

RN($\mathbf{K}_i$)	from φ, infer $\mathbf{K}_i\varphi$

Proof.

(i) φ hypothesis

(ii) $\mathbf{Kif}_i\varphi$ from (i) by RN($\mathbf{Kif}_i$)

(iii) $\varphi \wedge \mathbf{K}_i\varphi$ from Def2($\mathbf{K}_i$)

(iv) $\mathbf{K}_i\varphi$ from (iii)

□

K($\mathbf{K}_i$)	$\mathbf{K}_i(\varphi \to \psi) \to (\mathbf{K}_i\varphi \to \mathbf{K}_i\psi)$

Proof.

(i) $(\varphi \wedge \mathbf{Kif}_i\varphi \wedge (\varphi \to \psi) \wedge \mathbf{Kif}_i(\varphi \to \psi)) \to \mathbf{Kif}_i(\varphi \wedge (\varphi \to \psi))$ from `Conj`($\mathbf{Kif}_i$)

(ii) $(\varphi \wedge \mathbf{Kif}_i\varphi \wedge (\varphi \to \psi) \wedge \mathbf{Kif}_i(\varphi \to \psi)) \to \mathbf{Kif}_i(\varphi \wedge \psi)$ from (i) by `RE`($\mathbf{Kif}_i$)

(iii) $(\varphi \wedge \psi \wedge \mathbf{Kif}_i(\varphi \wedge \psi)) \to \mathbf{Kif}_i\psi$ from `Conj`($\mathbf{Kif}_i$)

(iv) $(\varphi \wedge \mathbf{Kif}_i\varphi \wedge (\varphi \to \psi) \wedge \mathbf{Kif}_i(\varphi \to \psi)) \to (\psi \wedge \mathbf{Kif}_i\psi)$ from (ii), (iii)

(v) $\mathbf{K}_i(\varphi \to \psi) \to (\mathbf{K}_i\varphi \to \mathbf{K}_i\psi)$ from (iv) by `Def2`($\mathbf{K}_i$)

□

`T`($\mathbf{K}_i$) $\mathbf{K}_i\varphi \to \varphi$

Proof.

(i) $(\varphi \wedge \mathbf{Kif}_\varphi) \to \varphi$

(ii) $\mathbf{K}_i\varphi \to \varphi$ from (i) by `Def2`($\mathbf{K}_i$)

□

$*$`5`($\mathbf{K}_i$) $\neg\mathbf{K}_i\varphi \to \mathbf{K}_i\neg\mathbf{K}_i\varphi$

Proof.

(i) $\mathbf{Kif}_i(\varphi \wedge \mathbf{Kif}_i\varphi)$ `45`$_2$($\mathbf{Kif}_i$)

(ii) $\mathbf{Kif}_i\mathbf{K}_i\varphi$ from (i) by `Def2`($\mathbf{K}_i$)

(iii) $\mathbf{Kif}_i\neg\mathbf{K}_i\varphi$ from (ii) by `Sym`($\mathbf{Kif}_i$)

(iv) $\neg\mathbf{K}_i\varphi \to (\neg\mathbf{K}_i\varphi \wedge \mathbf{Kif}_i\neg\mathbf{K}_i\varphi)$ from (iii)

(v) $\neg\mathbf{K}_i\varphi \to \mathbf{K}_i\neg\mathbf{K}_i\varphi$ from (iv) by `Def2`($\mathbf{K}_i$)

□

`Def`($\mathbf{E}$) $\mathbf{E}\varphi \leftrightarrow \bigwedge_{i\in Agt}\mathbf{K}_i\varphi$

Proof.

(i) $(\varphi \wedge \mathbf{Eif}\varphi) \leftrightarrow (\varphi \wedge \bigwedge_{i\in Agt}\mathbf{Kif}_i\varphi)$ from `Def2`($\mathbf{Eif}$)

(ii) $(\varphi \wedge \mathbf{Eif}\varphi) \leftrightarrow \bigwedge_{i\in Agt}(\varphi \wedge \mathbf{Kif}_i\varphi)$ from (i)

(iii) $\mathbf{E}\varphi \leftrightarrow \bigwedge_{i\in Agt}\mathbf{K}_i\varphi$ from (ii) by `Def2`($\mathbf{E}$), `Def2`($\mathbf{K}_i$)

□

`RN`($\mathbf{C}$) from φ, infer $\mathbf{C}\varphi$

Proof. Follow the lines of that of `RN`($\mathbf{K}_i$). □

`K`($\mathbf{C}$) $\mathbf{C}(\varphi \to \psi) \to (\mathbf{C}\varphi \to \mathbf{C}\psi)$

Proof. Follow the lines of that of `K`($\mathbf{K}_i$). □

$\texttt{T}(\mathbf{C})$	$\mathbf{C}\varphi \to \varphi$

Proof. Follow the lines of that of $\texttt{T}(\mathbf{K}_i)$. □

$\texttt{FP0}$	$\mathbf{C}\varphi \to \mathbf{E}\varphi$

Proof.

(i) $(\varphi \wedge \mathbf{Cif}\varphi) \to (\varphi \wedge \mathbf{Eif}\varphi)$ from $\texttt{GFP2}$

(ii) $\mathbf{C}\varphi \to \mathbf{E}\varphi$ from (i) by $\texttt{Def2}(\mathbf{C})$, $\texttt{Def2}(\mathbf{E})$

□

$\texttt{GFP0}$	$\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$

Proof.

(i) $(\mathbf{Eif}\varphi \wedge \mathbf{CifEif}\varphi) \to \mathbf{Cif}\varphi$ from $\texttt{GFP2}$

(ii) $\big((\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \wedge \mathbf{Cif}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi)\big) \to \mathbf{Cif}\varphi$
from (i) by thm. $\texttt{Def1}(\mathbf{Eif})$ and $\texttt{RE}(\mathbf{Cif})$

(iii) $\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$ from (ii) by $\texttt{Def2}(\mathbf{C})$, thm. $\texttt{Def1}(\mathbf{Cif})$

□

B Proof of Proposition 6.1

Proposition 6.1 *In the GFP-based axiomatics for KT,* $*5(\mathbf{C})$ *and the formula* $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$ *are interderivable.*

Proof. From the GFP-based axiomatics for KT and $*5(\mathbf{C})$ (recall that $4(\mathbf{C})$ is derivable from $\texttt{FP}'$, $\texttt{RN}(\mathbf{C})$ and $\texttt{GFP}$):

(i) $\mathbf{CC}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\varphi \to \mathbf{C}\varphi)$ from $\texttt{GFP}$, $\texttt{RN}(\mathbf{C})$, $\texttt{K}(\mathbf{C})$

(ii) $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{CC}(\varphi \to \mathbf{E}\varphi)$ $4(\mathbf{C})$

(iii) $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{CC}(\neg\mathbf{C}\varphi \to \neg\varphi)$ from (i), (ii) and $4(\mathbf{C})$

(iv) $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\mathbf{C}\neg\mathbf{C}\varphi \to \mathbf{C}\neg\varphi)$ from (iii) and $\texttt{K}(\mathbf{C})$

(v) $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\mathbf{C}\varphi \to \mathbf{C}\neg\varphi)$ from (iv) and $*5(\mathbf{C})$

(vi) $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$ from (v), $\texttt{FP}'$ and $\texttt{T}(\mathbf{C})$

From the GFP-based axiomatics for KT and $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$:

(i) $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$ hypothesis

(ii) $\mathbf{C}(\mathbf{C}\varphi \to \mathbf{EC}\varphi)$ from $\texttt{FP}'$ and $\texttt{RN}(\mathbf{C})$

(iii) $\mathbf{C}(\neg\mathbf{C}\varphi \to \mathbf{E}\neg\mathbf{C}\varphi)$ from (ii) and (i)

(iv) $\neg\mathbf{C}\varphi \to \mathbf{C}\neg\mathbf{C}\varphi$ from (iii) and $\texttt{GFP}$

□

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