

Temporal interpretation of intuitionistic quantifiers

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Abstract

We show that intuitionistic quantifiers admit the following temporal interpretation: $\forall xA$ is true at a world w iff A is true at every object in the domain of every future world, and $\exists xA$ is true at w iff A is true at some object in the domain of some past world. For this purpose we work with a predicate version of the well-known tense propositional logic $S4.t$. The predicate logic $Q^\circ S4.t$ is obtained by weakening the axioms of the standard predicate extension $QS4.t$ of $S4.t$ along the lines Corsi weakened QK to $Q^\circ K$. The Gödel translation embeds the predicate intuitionistic logic IQC into $QS4$ fully and faithfully. We provide a temporal version of the Gödel translation and prove that it embeds IQC into $Q^\circ S4.t$ fully and faithfully; that is, we show that a sentence is provable in IQC iff its translation is provable in $Q^\circ S4.t$. Faithfulness is proved using syntactic methods, while we prove fullness utilizing the generalized Kripke semantics of Corsi.

Keywords: Intuitionistic quantifiers, temporal interpretation, Gödel translation.

1 Introduction

Unlike classical connectives, intuitionistic connectives lack symmetry. It was noted already by McKinsey and Tarski [17] that Heyting algebras (which are algebraic models of intuitionistic propositional calculus IPC) are not symmetric even in the weak sense, meaning that the order-dual of a Heyting algebra may no longer be a Heyting algebra. In contrast, Boolean algebras (which are algebraic models of classical propositional calculus) are symmetric in the strong sense, meaning that the order-dual of a Boolean algebra A is not only a Boolean algebra, but even isomorphic to A .

This non-symmetry has been addressed by several authors, resulting in the concepts of bi-Heyting algebras and symmetric Heyting algebras. Bi-Heyting

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algebras are obtained by adding to the signature of Heyting algebras a binary operation of co-implication, while symmetric Heyting algebras by adding a de Morgan negation (and then co-implication becomes de Morgan dual of implication). The order-dual of a bi-Heyting algebra is again a bi-Heyting algebra, and the order-dual of a symmetric Heyting algebra A is even isomorphic to A . Thus, the class of bi-Heyting algebras is symmetric in the weak sense, while the class of symmetric Heyting algebras is symmetric in the strong sense (hence the name).

The Gödel translation of IPC into **S4** extends to a translation of the Heyting-Brouwer calculus **HB** of Rauszer [18] into the tense extension **S4.t** of **S4**, which has the future **S4**-modality \Box_F and the past **S4**-modality \Box_P . The algebraic models of **HB** are bi-Heyting algebras, and implication is interpreted using \Box_F and co-implication using \Box_P .

This story of non-symmetry also extends to intuitionistic quantifiers. Let **IQC** be the intuitionistic predicate calculus and **QS4** the predicate **S4**. Not only the intuitionistic quantifiers $\forall x$ and $\exists x$ are not definable from each other (unlike the classical quantifiers), but the Gödel translation $(\)^t$ of **IQC** into **QS4** is asymmetric in that $(\forall xA)^t = \Box\forall xA^t$ and $(\exists xA)^t = \exists xA^t$. This is manifested in the interpretation of intuitionistic quantifiers in Kripke models. Indeed, a world w of a Kripke model satisfies $\forall xA$ iff A is true at every object of the domain D_v of every world v accessible from w , while w satisfies $\exists xA$ iff A is true at some object in the domain D_w of w . If we think of the worlds of a Kripke model as “states of knowledge,” and the order between the states is temporal, then we can interpret the intuitionistic universal quantifier as “for every object in the future,” while the existential quantifier as “for some object in the present.”

In this article we present a more symmetric interpretation of intuitionistic quantifiers as “for every object in the future” for $\forall x$ and “for some object in the past” for $\exists x$. We show that such interpretation is supported by translating **IQC** fully and faithfully into a predicate tense logic by an appropriate modification of the Gödel translation. As far as we know, this approach has not been considered in the past. One obvious obstacle is that it is unclear what predicate tense logic to choose for such a translation. Indeed, a natural candidate would be the standard predicate extension **QS4.t** of **S4.t**. However, since **QS4.t** proves the Barcan formula, and hence the Kripke frames validating **QS4.t** have constant domains, **IQC** does not translate fully into **QS4.t**. Instead we work with a weaker logic in which the universal instantiation axiom

$$\forall xA \rightarrow A(y/x)$$

is replaced by a weaker version

$$\forall y(\forall xA \rightarrow A(y/x)).$$

This approach is along the lines of Kripke [15], Hughes and Cresswell [13], Fitting and Mendelsohn [6], and Corsi [3] who considered modal predicate logics

without the Barcan and/or converse Barcan formulas. The generalized Kripke frames considered in this semantics have two domains associated to each world, an inner domain and an outer domain. The inner domains are always contained in the outer domains and are not necessarily increasing. While variables are interpreted in the outer domains, the scope of quantifiers is restricted to the inner domains.

Utilizing this approach, we define a tense predicate logic $Q^\circ S4.t$ which is sound with respect to the generalized Kripke semantics with nonempty increasing inner domains and constant outer domains. We modify the Gödel translation to define a temporal translation of IQC into $Q^\circ S4.t$ as follows:

$$\begin{aligned}
 \perp^t &= \perp \\
 P(x_1, \dots, x_n)^t &= \Box_F P(x_1, \dots, x_n) && \text{for each } n\text{-ary predicate symbol } P \\
 (A \wedge B)^t &= A^t \wedge B^t \\
 (A \vee B)^t &= A^t \vee B^t \\
 (A \rightarrow B)^t &= \Box_F (A^t \rightarrow B^t) \\
 (\forall x A)^t &= \Box_F \forall x A^t \\
 (\exists x A)^t &= \Diamond_P \exists x A^t
 \end{aligned}$$

Here \Box_F is the **S4**-modality interpreted as “always in the future” and \Diamond_P is the **S4**-modality interpreted as “sometime in the past.” Thus, the modification of the Gödel translation concerns the clause for $\exists x A$. Our main result states that this translation is full and faithful in the following sense:

Main Theorem.

- For any formula A in the language of IQC, we have

$$\text{IQC} \vdash A \quad \text{iff} \quad Q^\circ S4.t \vdash \forall x_1 \dots \forall x_n A^t$$

where x_1, \dots, x_n are the free variables in A .

- If A is a sentence, then

$$\text{IQC} \vdash A \quad \text{iff} \quad Q^\circ S4.t \vdash A^t.$$

The proof of this surprising result is along the lines of the standard proof of fullness and faithfulness of the Gödel translation of IQC into **QS4**. We would like to stress that the main challenge is not so much the proof itself, but rather finding the “right” predicate tense modal logic into which to translate IQC. We find it of interest to explore philosophical (as well as practical) consequences of this new temporal point of view on IQC.

The paper is structured as follows. In Section 2 we recall the intuitionistic predicate logic IQC and its Kripke completeness. In Section 3 we briefly review the basics of modal predicate logics and their Kripke semantics, including weaker modal predicate logics. In Section 4 we recall the tense propositional logic **S4.t**, consider its standard predicate extension **QS4.t**, and then introduce

its weakening $\mathsf{Q}^\circ\mathsf{S4.t}$ which is our main tense predicate logic of interest. We conclude the section by observing that $\mathsf{Q}^\circ\mathsf{S4.t}$ is sound with respect to a version of the generalized Kripke semantics studied by Kripke [15], Hughes and Cresswell [13], Fitting and Mendelsohn [6], and Corsi [3]. Our main result, that IQC embeds into $\mathsf{Q}^\circ\mathsf{S4.t}$ fully and faithfully, is proved in Section 5. We prove faithfulness syntactically, while fullness is proved semantically. We conclude the paper with Section 6 in which we describe some open problems our study has generated. Finally, the Appendix contains the proofs of some technical lemmas used in Sections 4 and 5.

2 The intuitionistic predicate logic

Let IQC be the intuitionistic predicate logic. We recall that the language \mathcal{L} of IQC consists of countably many individual variables x, y, \dots , countably many n -ary predicate symbols P, Q, \dots (for each $n \geq 0$), the logical connectives $\perp, \wedge, \vee, \rightarrow$, and the quantifiers \forall, \exists . We do not add any constants to \mathcal{L} since this results in the temporal translation not being faithful (see Remark 5.11).

Formulas are defined as usual by induction and are denoted with upper case letters A, B, \dots . Let x, y be individual variables and A a formula. If x is a free variable of A and does not occur in the scope of $\forall y$ or $\exists y$, then we denote by $A(y/x)$ the formula obtained from A by replacing all the free occurrences of x by y .

The following definition of IQC is taken from [9, Sec 2.6]. We point out that, unlike [9], we prefer to work with axiom schemes, and hence do not need the inference rule of substitution.

Definition 2.1 The intuitionistic predicate logic IQC is the least set of formulas of \mathcal{L} containing all substitution instances of theorems of IPC, the axiom schemes

- (i) $\forall x A \rightarrow A(y/x)$ Universal instantiation (UI)
- (ii) $A(y/x) \rightarrow \exists x A$
- (iii) $\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall x B)$ with x not free in A
- (iv) $\forall x(A \rightarrow B) \rightarrow (\exists x A \rightarrow B)$ with x not free in B

and closed under the inference rules

$$\frac{A \quad A \rightarrow B}{B} \quad \text{Modus Ponens (MP)} \qquad \frac{A}{\forall x A} \quad \text{Generalization (Gen)}$$

We next describe Kripke semantics for IQC (see [16,8]).

Definition 2.2 An IQC-frame is a triple $\mathfrak{F} = (W, R, D)$ where

- W is a nonempty set whose elements are called the *worlds* of \mathfrak{F} .
- R is a partial order on W .

- D is a function that associates to each $w \in W$ a nonempty set D_w such that wRv implies $D_w \subseteq D_v$ for each $w, v \in W$. The set D_w is called the *domain* of w .

Definition 2.3

- An *interpretation* of \mathcal{L} in \mathfrak{F} is a function I associating to each world w and any n -ary predicate symbol P an n -ary relation $I_w(P) \subseteq (D_w)^n$ such that wRv implies $I_w(P) \subseteq I_v(P)$.
- A *model* is a pair $\mathfrak{M} = (\mathfrak{F}, I)$ where \mathfrak{F} is an IQC-frame and I is an interpretation in \mathfrak{F} .
- Let w be a world of \mathfrak{F} . A *w-assignment* is a function σ associating to each individual variable x an element $\sigma(x)$ of D_w . Note that if wRv , then σ is also a v -assignment.
- Let σ and τ be two w -assignments and x an individual variable. Then τ is said to be an *x -variant* of σ if $\tau(y) = \sigma(y)$ for all $y \neq x$.

We next recall the definition of when a formula A is true in a world w of a model $\mathfrak{M} = (\mathfrak{F}, I)$ under the w -assignment σ , written $\mathfrak{M} \models_w^\sigma A$.

Definition 2.4

$\mathfrak{M} \models_w^\sigma \perp$	never
$\mathfrak{M} \models_w^\sigma P(x_1, \dots, x_n)$	iff $(\sigma(x_1), \dots, \sigma(x_n)) \in I_w(P)$
$\mathfrak{M} \models_w^\sigma B \wedge C$	iff $\mathfrak{M} \models_w^\sigma B$ and $\mathfrak{M} \models_w^\sigma C$
$\mathfrak{M} \models_w^\sigma B \vee C$	iff $\mathfrak{M} \models_w^\sigma B$ or $\mathfrak{M} \models_w^\sigma C$
$\mathfrak{M} \models_w^\sigma B \rightarrow C$	iff for all v with wRv , if $\mathfrak{M} \models_v^\sigma B$, then $\mathfrak{M} \models_v^\sigma C$
$\mathfrak{M} \models_w^\sigma \forall x B$	iff for all v with wRv and each v -assignment τ that is an x -variant of σ , $\mathfrak{M} \models_v^\tau B$
$\mathfrak{M} \models_w^\sigma \exists x B$	iff there exists a w -assignment τ that is an x -variant of σ such that $\mathfrak{M} \models_w^\tau B$

Definition 2.5

- We say that A is *true* in a world w of \mathfrak{M} , written $\mathfrak{M} \models_w A$, if for all w -assignments σ , we have $\mathfrak{M} \models_w^\sigma A$.
- We say that A is *true* in \mathfrak{M} , written $\mathfrak{M} \models A$, if for all worlds $w \in W$, we have $\mathfrak{M} \models_w A$.
- We say that A is *valid* in a frame \mathfrak{F} , written $\mathfrak{F} \models A$, if for all models \mathfrak{M} based on \mathfrak{F} , we have $\mathfrak{M} \models A$.

We have the following well-known completeness of IQC with respect to Kripke semantics.

Theorem 2.6 ([16]) *The intuitionistic predicate logic IQC is sound and complete with respect to Kripke semantics; that is, for each formula A ,*

$$\text{IQC} \vdash A \text{ iff } \mathfrak{F} \models A \text{ for each IQC-frame } \mathfrak{F}.$$

3 Modal predicate logics

Modal predicate logics were first studied by Barcan [1] and Carnap [2] in 1940s. The semantic study of modal predicate logics was initiated by Kripke [14,15] in late 1950s/early 1960s. Since then many completeness results have been obtained with respect to Kripke semantics, but there is also a large body of incompleteness results, which is one of the reasons that the model theory of modal predicate logics is less advanced than that of modal propositional logics (see, e.g., [9,10] and the references therein).

Let \mathbf{K} be the least normal modal propositional logic and let \mathbf{QK} be the standard predicate extension of \mathbf{K} . The language \mathcal{L}_\square of \mathbf{QK} is the extension of \mathcal{L} with the modality \square . Since the modal logics we consider are based on the classical logic, it is sufficient to only consider the logical connectives \perp, \rightarrow and the quantifier \forall . The logical connectives $\wedge, \vee, \neg, \leftrightarrow$, the quantifier \exists , and the modality \diamond are treated as usual abbreviations.

We next recall the definition of \mathbf{QK} (see, e.g., [9, Sec 2.6], but note, as in Section 2, that we work with axiom schemes instead of having the inference rule of substitution).

Definition 3.1 The modal predicate logic \mathbf{QK} is the least set of formulas of \mathcal{L}_\square containing all substitution instances of theorems of \mathbf{K} , the axiom schemes (i) and (iii) of Definition 2.1, and closed under (MP), (Gen), and

$$\frac{A}{\square A} \quad \text{Necessitation (N)}$$

The definition of \mathbf{QK} -frames $\mathfrak{F} = (W, R, D)$ is the same as that of IQC-frames (see Definition 2.2) with the only difference that R can be an arbitrary relation. *Models* are also defined the same way, but without the requirement that wRv implies $I_w(P) \subseteq I_v(P)$. The connectives and quantifiers are interpreted at each world in the usual classical way, and

$$\mathfrak{M} \models_w^\sigma \square A \text{ iff } (\forall v \in W)(wRv \Rightarrow \mathfrak{M} \models_v^\sigma A).$$

Truth and *validity* of formulas are defined as usual.

We next give a brief history of first Kripke completeness results for modal predicate logics. In 1959 Kripke [14] proved Kripke completeness of predicate S5. In late 1960s Cresswell [4,5] (see also Hughes and Cresswell [12]), Schütte [19], and Thomason [20] proved Kripke completeness of predicate T and S4. Kripke completeness of \mathbf{QK} was first established by Gabbay [7, Thm. 8.5]²:

² We would like to thank Ilya Shapirovsky and Valentin Shehtman for useful discussions on the history of Kripke completeness for modal predicate logics.

Theorem 3.2 *The modal predicate logic QK is sound and complete with respect to Kripke semantics.*

The following two principles play an important role in the study of modal predicate logics. They were first considered by Barcan [1].

$$\begin{array}{ll} \Box\forall xA \rightarrow \forall x\Box A & \text{converse Barcan formula} \quad (\text{CBF}) \\ \forall x\Box A \rightarrow \Box\forall xA & \text{Barcan formula} \quad (\text{BF}) \end{array}$$

It is easy to see that CBF is a theorem of QK. Indeed, this follows from Theorem 3.2 and the fact that domains of each QK-frame are increasing. On the other hand, a QK-frame validates BF iff it has *constant domains*, meaning that wRv implies $D_w = D_v$, and we have the following well-known theorem (see, e.g., [7, Thm. 9.3]):

Theorem 3.3 *The logic QK + BF is sound and complete with respect to the class of QK-frames with constant domains.*

A modal predicate logic whose Kripke frames have neither increasing nor decreasing domains was considered already by Kripke [15]. Building on this work, Hughes and Cresswell [13, pp. 304–309] introduced a similar predicate modal logic and proved its completeness with respect to a generalized Kripke semantics. Fitting and Mendelsohn [6, Sec. 6.2] gave an alternate axiomatization of this logic. Building on the work of Fitting and Mendelsohn, Corsi [3] defined the system $Q^\circ K$ whose axiomatization contains a weakening of the universal instantiation axiom.

Definition 3.4 The logic $Q^\circ K$ is the least set of formulas of \mathcal{L}_\Box containing all substitution instances of theorems of K, the axiom schemes

- (i) $\forall y(\forall xA \rightarrow A(y/x))$ (UI $^\circ$)
- (ii) $\forall x(A \rightarrow B) \rightarrow (\forall xA \rightarrow \forall xB)$
- (iii) $\forall x\forall yA \leftrightarrow \forall y\forall xA$
- (iv) $A \rightarrow \forall xA$ with x not free in A

and closed under (MP), (Gen), and (N).

Remark 3.5 In Definition 3.4, replacing UI $^\circ$ with UI yields an equivalent definition of QK. Therefore, $Q^\circ K$ is contained in QK.

Kripke frames for $Q^\circ K$ generalize Kripke frames for QK by having two domains, inner and outer.

Definition 3.6 A $Q^\circ K$ -frame is a quadruple $\mathfrak{F} = (W, R, D, U)$ where

- (W, R) is a K-frame.
- D is a function that associates to each $w \in W$ a set D_w . The set D_w is called the *inner domain* of w .

- U is a nonempty set containing the union of all the D_w . The set U is called the *outer domain* of \mathfrak{F} .

Definition 3.6 is a particular case of the frames considered by Corsi [3] where increasing outer domains are allowed. For our purposes, taking a fixed outer domain U is sufficient. We recall from [3] how to interpret \mathcal{L}_\square in a $\mathbb{Q}^\circ\mathbb{K}$ -frame $\mathfrak{F} = (W, R, D, U)$.

Definition 3.7

- An *interpretation* of \mathcal{L}_\square in \mathfrak{F} is a function I associating to each world w and an n -ary predicate symbol P an n -ary relation $I_w(P) \subseteq U^n$.
- A *model* is a pair $\mathfrak{M} = (\mathfrak{F}, I)$ where \mathfrak{F} is a $\mathbb{Q}^\circ\mathbb{K}$ -frame and I is an interpretation in \mathfrak{F} .
- An *assignment* in \mathfrak{F} is a function σ that associates to each individual variable an element of U .
- If σ and τ are two assignments and x is an individual variable, τ is said to be an x -variant of σ if $\tau(y) = \sigma(y)$ for all $y \neq x$.
- We say that an assignment σ is w -inner for $w \in W$ if $\sigma(x) \in D_w$ for each individual variable x .

We next recall from [3] the definition of when a formula A is true in a world w of a model $\mathfrak{M} = (\mathfrak{F}, I)$ under the assignment σ , written $\mathfrak{M} \models_w^\sigma A$.

Definition 3.8

$$\begin{array}{ll} \mathfrak{M} \models_w^\sigma \perp & \text{never} \\ \mathfrak{M} \models_w^\sigma P(x_1, \dots, x_n) & \text{iff } (\sigma(x_1), \dots, \sigma(x_n)) \in I_w(P) \\ \mathfrak{M} \models_w^\sigma B \rightarrow C & \text{iff } \mathfrak{M} \models_w^\sigma B \text{ implies } \mathfrak{M} \models_w^\sigma C \\ \mathfrak{M} \models_w^\sigma \forall x B & \text{iff for all } x\text{-variants } \tau \text{ of } \sigma \text{ with } \tau(x) \in D_w, \mathfrak{M} \models_w^\tau B \\ \mathfrak{M} \models_w^\sigma \Box B & \text{iff for all } v \text{ such that } wRv, \mathfrak{M} \models_v^\sigma B \end{array}$$

Definition 3.9 A formula A is *true* in a model $\mathfrak{M} = (\mathfrak{F}, I)$ at the world $w \in W$ (in symbols $\mathfrak{M} \models_w A$) if for all assignments σ , we have $\mathfrak{M} \models_w^\sigma A$. The definition of *truth* in a model and *validity* in a frame are the same as in Definition 2.5.

We have the following completeness result for $\mathbb{Q}^\circ\mathbb{K}$, see [3, Thm. 1.32] and its proof.

Theorem 3.10 $\mathbb{Q}^\circ\mathbb{K}$ is sound and complete with respect to the class of $\mathbb{Q}^\circ\mathbb{K}$ -frames.

Definition 3.11 Let $\mathfrak{F} = (W, R, D, U)$ be a $\mathbb{Q}^\circ\mathbb{K}$ -frame.

- We say that \mathfrak{F} has *increasing inner domains* if wRv implies $D_w \subseteq D_v$ for each $w, v \in W$.

- We say that \mathfrak{F} has *decreasing inner domains* if wRv implies $D_v \subseteq D_w$ for each $w, v \in W$.
- If \mathfrak{F} has both increasing and decreasing inner domains, we say that it has *constant inner domains*.

The following axiom scheme guarantees nonempty inner domains (hence the abbreviation):

$$\forall xA \rightarrow A \text{ with } x \text{ not free in } A \quad (\text{NID})$$

The next proposition is not difficult to verify (see, e.g., [6, Sec. 4.9] and [3, pp. 1487–1488]).

Proposition 3.12 *Let $\mathfrak{F} = (W, R, D, U)$ be a Q°K -frame.*

- \mathfrak{F} validates CBF iff \mathfrak{F} has increasing inner domains.
- \mathfrak{F} validates BF iff \mathfrak{F} has decreasing inner domains.
- \mathfrak{F} validates NID iff \mathfrak{F} has nonempty inner domains.

We have the following completeness results for logics obtained by adding CBF, BF, and NID to Q°K (see [3, Thms. 1.30, 1.32, and Footnote 7]):

Theorem 3.13

- $\text{Q}^\circ\text{K} + \text{CBF}$ is sound and complete with respect to the class of Q°K -frames with increasing inner domains.
- $\text{Q}^\circ\text{K} + \text{CBF} + \text{BF}$ is sound and complete with respect to the class of Q°K -frames with constant inner domains.
- Adding NID to the above two logics or to Q°K yields completeness of the resulting logics with respect to the corresponding classes of frames which have nonempty inner domains.

On the other hand, completeness of $\text{Q}^\circ\text{K} + \text{BF}$ remains open (see [3, p. 1510]).

4 The logic $\text{Q}^\circ\text{S4.t}$

The tense predicate logic we will translate IQC into is based on the well-known tense propositional logic S4.t . We use \Box_F (“always in the future”) and \Box_P (“always in the past”) as temporal modalities. Then \Diamond_F (“sometime in the future”) and \Diamond_P (“sometime in the past”) are usual abbreviations $\neg\Box_F\neg$ and $\neg\Box_P\neg$.

Definition 4.1 The logic S4.t is the least set of formulas of the tense propositional language containing all substitution instances of S4 -axioms for both \Box_F and \Box_P , the axiom schemes

- (i) $A \rightarrow \Box_P\Diamond_F A$

(ii) $A \rightarrow \Box_F \Diamond_P A$

and closed under (MP) and

$$\frac{A}{\Box_F A} \quad \Box_F\text{-Necessitation (N}_F\text{)} \qquad \frac{A}{\Box_P A} \quad \Box_P\text{-Necessitation (N}_P\text{)}$$

Relational semantics of **S4.t** consists of Kripke frames $\mathfrak{F} = (W, R)$ where R is reflexive and transitive. As usual, propositional letters are interpreted as subsets of W , classical connectives as the corresponding set-theoretic operations on the powerset of W , and for temporal modalities we set:

$$\begin{aligned} w \models \Box_F A & \text{ iff } (\forall v \in W)(wRv \Rightarrow v \models A) \\ w \models \Box_P A & \text{ iff } (\forall v \in W)(vRw \Rightarrow v \models A) \end{aligned}$$

It is well known that **S4.t** is sound and complete with respect to its relational semantics.

Let \mathcal{L}_T be the bimodal predicate language obtained by extending \mathcal{L} with two modalities \Box_F and \Box_P .

Definition 4.2 The logic **QS4.t** is the least set of formulas of \mathcal{L}_T containing all substitution instances of theorems of **S4.t**, the axiom schemes (i) and (iii) of Definition 2.1, and closed under (MP), (Gen), (N_F), and (N_P).

The following are temporal versions of CBF and BF:

$$\begin{array}{lll} \Box_F \forall x A \rightarrow \forall x \Box_F A & \text{converse Barcan formula for } \Box_F & (\text{CBF}_F) \\ \forall x \Box_F A \rightarrow \Box_F \forall x A & \text{Barcan formula for } \Box_F & (\text{BF}_F) \\ \Box_P \forall x A \rightarrow \forall x \Box_P A & \text{converse Barcan formula for } \Box_P & (\text{CBF}_P) \\ \forall x \Box_P A \rightarrow \Box_P \forall x A & \text{Barcan formula for } \Box_P & (\text{BF}_P) \end{array}$$

The proof that $\text{QK} \vdash \text{CBF}$ (see, e.g., [15, p. 88]) can be adapted to prove that $\text{QS4.t} \vdash \text{CBF}_F$ and $\text{QS4.t} \vdash \text{CBF}_P$. It is also well known that CBF_F and BF_P , as well as CBF_P and BF_F are derivable from each other in any tense predicate logic. Therefore, all four are theorems of **QS4.t**. This is reflected in the fact that **QS4.t**-frames have constant domains. Indeed, **QS4.t** is complete with respect to this semantics (see Section 6). But this is problematic for translating **IQC** fully into **QS4.t** since **IQC**-frames with constant domains validate the additional axiom $\forall x(A \vee B) \rightarrow (A \vee \forall x B)$, where x is not free in A , which is not a theorem of **IQC** (see, e.g., [8, p. 53, Cor. 8]).

Consequently, we need to work with a weaker logic than **QS4.t**. To this end, we introduce the logic **Q°S4.t**, which weakens **QS4.t** the same way **Q°K** weakens **QK**.

Definition 4.3 The logic **Q°S4.t** is the least set of formulas of \mathcal{L}_T containing all substitution instances of theorems of **S4.t**, the axiom schemes (i), (ii), (iii), (iv) of **Q°K** (see Definition 3.4), **NID**, CBF_F , and closed under (MP), (Gen), (N_F), and (N_P).

As follows from Proposition A.1 in the Appendix, BF_P is a theorem of $\text{Q}^\circ\text{S4.t}$. In fact, CBF_F and BF_P are derivable from each other and the other axioms of $\text{Q}^\circ\text{S4.t}$.

Definition 4.4 A $\text{Q}^\circ\text{S4.t}$ -frame is a Q°K -frame $\mathfrak{F} = (W, R, D, U)$ (see Definition 3.6) with nonempty increasing inner domains whose accessibility relation is reflexive and transitive.

Models and assignments are defined as in Definition 3.7. The clauses of when a formula A of \mathcal{L}_T is true in a world w of a $\text{Q}^\circ\text{S4.t}$ -model $\mathfrak{M} = (\mathfrak{F}, I)$ under the assignment σ , written $\mathfrak{M} \models_w^\sigma A$, are defined as in Definition 3.8, but we replace the \Box -clause with the following two clauses:

$$\begin{aligned} \mathfrak{M} \models_w^\sigma \Box_F B & \text{ iff } (\forall v \in W)(wRv \Rightarrow \mathfrak{M} \models_v^\sigma B) \\ \mathfrak{M} \models_w^\sigma \Box_P B & \text{ iff } (\forall v \in W)(vRw \Rightarrow \mathfrak{M} \models_v^\sigma B) \end{aligned}$$

For formulas of \mathcal{L}_T we define truth in a model and validity in a frame as in Definition 3.9.

Theorem 4.5 $\text{Q}^\circ\text{S4.t}$ is sound with respect to the class of $\text{Q}^\circ\text{S4.t}$ -frames; that is, for each formula A of \mathcal{L}_T and $\text{Q}^\circ\text{S4.t}$ -frame \mathfrak{F} , from $\text{Q}^\circ\text{S4.t} \vdash A$ it follows that $\mathfrak{F} \models A$.

Proof. It is sufficient to show that each axiom scheme is valid in all $\text{Q}^\circ\text{S4.t}$ -frames and that each rule of inference preserves validity. This can be done by direct verification. We only show that the axiom scheme CBF_F is valid in all $\text{Q}^\circ\text{S4.t}$ -frames. Let $\mathfrak{M} = (\mathfrak{F}, I)$ be a $\text{Q}^\circ\text{S4.t}$ -model, $w \in W$, and σ an assignment. If $\mathfrak{M} \models_w^\sigma \Box_F \forall x A$, then for all v with wRv we have $\mathfrak{M} \models_v^\sigma \forall x A$. This implies that for each x -variant τ of σ with $\tau(x) \in D_v$ we have $\mathfrak{M} \models_v^\tau A$. Since $D_w \subseteq D_v$, this is in particular true for x -variants τ of σ with $\tau(x) \in D_w$. Therefore, for each x -variant τ of σ with $\tau(x) \in D_w$ and for each v with wRv we have $\mathfrak{M} \models_v^\tau A$. Thus, for each x -variant τ of σ with $\tau(x) \in D_w$, we have $\mathfrak{M} \models_w^\sigma \Box_F A$. Consequently, $\mathfrak{M} \models_w^\sigma \forall x \Box_F A$. This shows that $\mathfrak{F} \models \Box_F \forall x A \rightarrow \forall x \Box_F A$ for each $\text{Q}^\circ\text{S4.t}$ -frame \mathfrak{F} . \square

On the other hand, completeness of $\text{Q}^\circ\text{S4.t}$ remains an interesting open problem, which is related to the open problem of completeness of $\text{Q}^\circ\text{K} + \text{BF}$ (see Section 6).

5 The translation

In this section we prove our main result that the temporal modification (described in the Introduction) of the Gödel translation embeds IQC into $\text{Q}^\circ\text{S4.t}$ fully and faithfully. Our strategy is to prove faithfulness of the translation syntactically, while fullness will be proved by semantical means, utilizing Kripke completeness of IQC.

Our syntactic proof of faithfulness is based on the following technical lemma, the proof of which we give in the Appendix. To keep the notation simple, we denote lists of variables by bold letters. If $\mathbf{x} = x_1, \dots, x_n$, we write $\forall \mathbf{x}$ for $\forall x_1 \dots \forall x_n$. We point out that it is a consequence of axioms (ii) and (iii) of Q°K that from the point of view of provability in $\text{Q}^\circ\text{S4.t}$, the order of variables in $\forall \mathbf{x}$ does not matter.

Lemma 5.1

- (i) Let C be an instance of an axiom scheme of IQC and \mathbf{x} the list of free variables in C . Then $\text{Q}^\circ\text{S4.t} \vdash \forall \mathbf{x} C^t$.
- (ii) Let A, B be formulas of \mathcal{L} , \mathbf{x} the list of variables free in $A \rightarrow B$, \mathbf{y} the list of variables free in A , and \mathbf{z} the list of variables free in B . If $\text{Q}^\circ\text{S4.t} \vdash \forall \mathbf{x}(A \rightarrow B)^t$ and $\text{Q}^\circ\text{S4.t} \vdash \forall \mathbf{y} A^t$, then $\text{Q}^\circ\text{S4.t} \vdash \forall \mathbf{z} B^t$.
- (iii) Let A be a formula of \mathcal{L} , x a variable, \mathbf{y} the list of variables free in A , and \mathbf{z} the list of variables free in $\forall x A$. If $\text{Q}^\circ\text{S4.t} \vdash \forall \mathbf{y} A^t$, then $\text{Q}^\circ\text{S4.t} \vdash \forall \mathbf{z} (\forall x A)^t$.

Proof. For (i) see the proof of Lemma A.5, for (ii) see the proof of Lemma A.6, and for (iii) see the proof of Lemma A.7. \square

Theorem 5.2 Let A be a formula of \mathcal{L} and x_1, \dots, x_n the free variables of A . If $\text{IQC} \vdash A$, then $\text{Q}^\circ\text{S4.t} \vdash \forall x_1 \dots \forall x_n A^t$.

Proof. The proof is by induction on the length of the proof of A in IQC. If A is an instance of an axiom of IQC, then the result follows from Lemma 5.1(i). Lemma 5.1(ii) takes care of the case in which the last step of the proof of A is an application of (MP). Finally, if the last step of the proof of A is an application of (Gen) to the variable x , use Lemma 5.1(iii). \square

Remark 5.3 We are prefixing the translation of A with $\forall x_1 \dots \forall x_n$ because it is not true in general that $\text{IQC} \vdash A$ implies $\text{Q}^\circ\text{S4.t} \vdash A^t$. For example, if A is an instance of the universal instantiation axiom, which is an axiom of IQC, then A^t is not in general a theorem of $\text{Q}^\circ\text{S4.t}$.

Definition 5.4

- For an IQC-frame $\mathfrak{F} = (W, R, D)$ let $\overline{\mathfrak{F}} = (W, R, D, U)$ where $U = \bigcup \{D_w \mid w \in W\}$.
- For an IQC-model $\mathfrak{M} = (\mathfrak{F}, I)$ let $\overline{\mathfrak{M}} = (\overline{\mathfrak{F}}, I)$.

Remark 5.5

- It is obvious that $\overline{\mathfrak{F}}$ is a $\text{Q}^\circ\text{S4.t}$ -frame.
- If I is an interpretation in \mathfrak{F} , then I is also an interpretation in $\overline{\mathfrak{F}}$ because for each n -ary predicate letter P we have $I_w(P) \subseteq D_w^n \subseteq U^n$. Therefore, $\overline{\mathfrak{M}}$ is well defined.

- The w -assignments in \mathfrak{F} are exactly the w -inner assignments in $\overline{\mathfrak{F}}$.

The proof of the following technical lemma is given in the Appendix.

Lemma 5.6 *If A is a formula of \mathcal{L} , then $\text{Q}^\circ\text{S4.t} \vdash A^t \rightarrow \Box_F A^t$.*

Proof. See the proof of Lemma A.2. \square

Lemma 5.7 *Let A be a formula of \mathcal{L} , $\mathfrak{M} = (\mathfrak{F}, I)$ a $\text{Q}^\circ\text{S4.t}$ -model, and σ an assignment in \mathfrak{F} . If $v, w \in W$ with vRw , then $\mathfrak{M} \models_v^\sigma A^t$ implies $\mathfrak{M} \models_w^\sigma A^t$.*

Proof. Suppose vRw and $\mathfrak{M} \models_v^\sigma A^t$. By Lemma 5.6 and Theorem 4.5, $\mathfrak{M} \models_v^\sigma A^t \rightarrow \Box_F A^t$. Therefore, $\mathfrak{M} \models_v^\sigma \Box_F A^t$, which yields $\mathfrak{M} \models_w^\sigma A^t$ because vRw . \square

Proposition 5.8 *Let A be a formula of \mathcal{L} , $\mathfrak{M} = (\mathfrak{F}, I)$ an IQC-model based on an IQC-frame $\mathfrak{F} = (W, R, D)$, and $w \in W$.*

- (i) *For each w -assignment σ ,*

$$\mathfrak{M} \models_w^\sigma A \text{ iff } \overline{\mathfrak{M}} \models_w^\sigma A^t.$$

- (ii) *If x_1, \dots, x_n are the free variables of A , then*

$$\mathfrak{M} \models_w A \text{ iff } \overline{\mathfrak{M}} \models_w \forall x_1 \dots \forall x_n A^t.$$

Proof. (i). Induction on the complexity of A . Let A be an atomic formula $P(x_1, \dots, x_n)$. Since wRv implies $I_w(P) \subseteq I_v(P)$ and R is reflexive, we have

$$\begin{aligned} \mathfrak{M} \models_w^\sigma P(x_1, \dots, x_n) &\text{ iff } (\sigma(x_1), \dots, \sigma(x_n)) \in I_w(P) \\ &\text{ iff } (\forall v \in W)(wRv \Rightarrow (\sigma(x_1), \dots, \sigma(x_n)) \in I_v(P)) \\ &\text{ iff } \overline{\mathfrak{M}} \models_w^\sigma \Box_F P(x_1, \dots, x_n) \\ &\text{ iff } \overline{\mathfrak{M}} \models_w^\sigma P(x_1, \dots, x_n)^t \end{aligned}$$

The cases where $A = \perp$, $A = B \wedge C$, and $A = B \vee C$ are straightforward.

If $A = B \rightarrow C$, then using the inductive hypothesis, we have

$$\begin{aligned} \mathfrak{M} \models_w^\sigma B \rightarrow C &\text{ iff } (\forall v \in W)(wRv \Rightarrow (\mathfrak{M} \models_v^\sigma B \Rightarrow \mathfrak{M} \models_v^\sigma C)) \\ &\text{ iff } (\forall v \in W)(wRv \Rightarrow (\overline{\mathfrak{M}} \models_v^\sigma B^t \Rightarrow \overline{\mathfrak{M}} \models_v^\sigma C^t)) \\ &\text{ iff } \overline{\mathfrak{M}} \models_w^\sigma \Box_F (B^t \rightarrow C^t) \\ &\text{ iff } \overline{\mathfrak{M}} \models_w^\sigma (B \rightarrow C)^t \end{aligned}$$

If $A = \forall xB$, then using the inductive hypothesis, we have

$$\begin{aligned}
\mathfrak{M} \models_w^\sigma \forall xB &\text{ iff } (\forall v \in W)(wRv \Rightarrow \text{for each } v\text{-assignment } \tau \text{ that is} \\
&\quad \text{an } x\text{-variant of } \sigma \text{ we have } \mathfrak{M} \models_v^\tau B) \\
&\text{ iff } (\forall v \in W)(wRv \Rightarrow \text{for each assignment } \tau \text{ that is} \\
&\quad \text{an } x\text{-variant of } \sigma \text{ with } \tau(x) \in D_v \text{ we have } \overline{\mathfrak{M}} \models_v^\tau B^t) \\
&\text{ iff } \overline{\mathfrak{M}} \models_w^\sigma \Box_F \forall xB^t \\
&\text{ iff } \overline{\mathfrak{M}} \models_w^\sigma (\forall xB)^t
\end{aligned}$$

If $A = \exists xB$, then using the inductive hypothesis, reflexivity of R , Lemma 5.7, and the fact that vRw implies $D_v \subseteq D_w$, we have

$$\begin{aligned}
\mathfrak{M} \models_w^\sigma \exists xB &\text{ iff there is a } w\text{-assignment } \tau \text{ that is an } x\text{-variant of } \sigma \\
&\quad \text{such that } \mathfrak{M} \models_w^\tau B \\
&\text{ iff there is an assignment } \tau \text{ that is an } x\text{-variant of } \sigma \\
&\quad \text{with } \tau(x) \in D_w \text{ such that } \overline{\mathfrak{M}} \models_w^\tau B^t \\
&\text{ iff there is } v \in W \text{ such that } vRw \text{ and an assignment } \rho \text{ that is} \\
&\quad \text{an } x\text{-variant of } \sigma \text{ with } \rho(x) \in D_v \text{ such that } \overline{\mathfrak{M}} \models_v^\rho B^t \\
&\text{ iff } \overline{\mathfrak{M}} \models_w^\sigma \Diamond_P \exists xB^t \\
&\text{ iff } \overline{\mathfrak{M}} \models_w^\sigma (\exists xB)^t
\end{aligned}$$

(ii). By Definition 2.5, $\mathfrak{M} \models_w A$ iff $\mathfrak{M} \models_w^\sigma A$ for each w -assignment σ . As noted in Remark 5.5, w -assignments in \mathfrak{F} are exactly the w -inner assignments in $\overline{\mathfrak{F}}$. Therefore, by (i), $\mathfrak{M} \models_w A$ iff $\overline{\mathfrak{M}} \models_w^\sigma A^t$ for each w -inner assignment σ . It follows from the interpretation of the universal quantifier in $\overline{\mathfrak{M}}$ that $\overline{\mathfrak{M}} \models_w^\sigma A^t$ for each w -inner assignment σ iff $\overline{\mathfrak{M}} \models_w \forall x_1 \cdots \forall x_n A^t$. Thus, $\mathfrak{M} \models_w A$ iff $\overline{\mathfrak{M}} \models_w \forall x_1 \cdots \forall x_n A^t$. \square

Theorem 5.9 *Let A be a formula of \mathcal{L} and x_1, \dots, x_n the free variables of A . If $\text{Q}^\circ\text{S4.t} \vdash \forall x_1 \cdots \forall x_n A^t$, then $\text{IQC} \vdash A$.*

Proof. Suppose $\text{IQC} \not\vdash A$. Theorem 2.6 implies that there is an IQC-model \mathfrak{M} such that $\mathfrak{M} \not\models_w A$ for some world w . By Proposition 5.8(ii), $\overline{\mathfrak{M}} \not\models_w \forall x_1 \cdots \forall x_n A^t$. Thus, $\text{Q}^\circ\text{S4.t} \not\vdash \forall x_1 \cdots \forall x_n A^t$ by Theorem 4.5. \square

By putting Theorems 5.2 and 5.9 together we arrive at the main result of the paper mentioned in the introduction.

Theorem 5.10

- *Let A be a formula of \mathcal{L} and x_1, \dots, x_n the free variables of A . We have*

$$\text{IQC} \vdash A \text{ iff } \text{Q}^\circ\text{S4.t} \vdash \forall x_1 \cdots \forall x_n A^t.$$

- If A is a sentence of \mathcal{L} , then

$$\text{IQC} \vdash A \text{ iff } \text{Q}^\circ\text{S4.t} \vdash A^t.$$

Remark 5.11 If we allow constants in \mathcal{L} , Theorem 5.9 is no longer true in its current form. Indeed, constants in IQC and $\text{Q}^\circ\text{S4.t}$ behave like free variables and we would have the problem described in Remark 5.3. However, it can be adjusted as follows. Let A be a formula containing free variables x_1, \dots, x_n and constants c_1, \dots, c_m . If $A(y_1/c_1, \dots, y_m/c_m)$ is the formula obtained by replacing all the constants with fresh variables y_1, \dots, y_m , then $\text{IQC} \vdash A$ iff $\text{Q}^\circ\text{S4.t} \vdash \forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_m A^t(y_1/c_1, \dots, y_m/c_m)$.

6 Open problems

As follows from Theorem 4.5, $\text{Q}^\circ\text{S4.t}$ is sound with respect to the class of $\text{Q}^\circ\text{S4.t}$ -frames. However, its completeness remains an interesting open problem. The standard Henkin construction was modified by Hughes and Cresswell [13] and Corsi [3] to obtain completeness of Q°K . If we adapt their technique to $\text{Q}^\circ\text{S4.t}$, we obtain two relations R_F and R_P on the canonical model, one coming from \Box_F and the other from \Box_P . There does not seem to be an obvious way to select an appropriate submodel in which the restrictions of these two relations are inverses of each other because the outer domains of accessible worlds are forced to increase by the construction. This problem disappears when constructing the canonical model for QS4.t because the presence of BF_F and CBF_P in each world allows us to select witnesses without expanding the domains of accessible worlds, thus yielding that QS4.t is sound and complete with respect to the class of QS4.t -frames.

The problem of completeness of $\text{Q}^\circ\text{S4.t}$ seems to be closely related to the open problem, stated in [3, p. 1510], of whether $\text{Q}^\circ\text{K} + \text{BF}$ is Kripke complete. It appears that answering one of these problems could also provide an answer to the other.

One of the reviewers pointed out that another natural direction is to study the intermediate predicate logics and the corresponding extensions of $\text{Q}^\circ\text{S4.t}$ for which our temporal translation remains full and faithful. Finally, it is worth investigating whether other tense predicate logics (such as the ones considered in [11]) could be used for translating IQC fully and faithfully. Some such systems admit presheaf semantics which is more general than Kripke semantics.

Appendix

A Additional facts needed in Sections 4 and 5

Proposition A.1 $\text{Q}^\circ\text{S4.t} \vdash \text{BF}_P$.

Proof. We first show that $\text{Q}^\circ\text{S4.t} \vdash \Diamond_F \forall x B \rightarrow \forall x \Diamond_F B$ for any formula B . We have the proof

1. $\forall x(\forall xB \rightarrow B)$
2. $\forall x\Box_F(\forall xB \rightarrow B)$
3. $\Box_F(\forall xB \rightarrow B) \rightarrow (\Diamond_F\forall xB \rightarrow \Diamond_FB)$
4. $\forall x\Box_F(\forall xB \rightarrow B) \rightarrow \forall x(\Diamond_F\forall xB \rightarrow \Diamond_FB)$
5. $\forall x(\Diamond_F\forall xB \rightarrow \Diamond_FB)$
6. $\forall x\Diamond_F\forall xB \rightarrow \forall x\Diamond_FB$
7. $\Diamond_F\forall xB \rightarrow \forall x\Diamond_FB$

Here 1 is an instance of UI° ; 2 is obtained from 1 by adding \Box_F inside $\forall x$ by applying (N_F) , CBF_F , and (MP) ; 3 is a substitution instance of the K -theorem $\Box_F(C \rightarrow D) \rightarrow (\Diamond_FC \rightarrow \Diamond_FD)$ for \Box_F ; 4 is obtained from 3 by first adding and then distributing $\forall x$ inside the implication by applying (Gen) , axiom (ii) of Q°K , and (MP) ; 5 follows from 2 and 4 by (MP) ; 6 is obtained from 5 by distributing $\forall x$; and 7 follows from 6 and axiom (iv) of Q°K .

We now prove $\forall x\Box_PA \rightarrow \Box_P\forall xA$.

1. $\forall x\Box_PA \rightarrow \Box_P\Diamond_F\forall x\Box_PA$
2. $\Diamond_F\forall x\Box_PA \rightarrow \forall x\Diamond_F\Box_PA$
3. $\Box_P\Diamond_F\forall x\Box_PA \rightarrow \Box_P\forall x\Diamond_F\Box_PA$
4. $\Diamond_F\Box_PA \rightarrow A$
5. $\forall x\Diamond_F\Box_PA \rightarrow \forall xA$
6. $\Box_P\forall x\Diamond_F\Box_PA \rightarrow \Box_P\forall xA$
7. $\forall x\Box_PA \rightarrow \Box_P\forall xA$

Here 1 is an instance of axiom (i) of S4.t ; 2 is an instance of $\Diamond_F\forall xB \rightarrow \forall x\Diamond_FB$ proved above; 3 and 6 follow from 2 and 5 by adding and distributing \Box_P in the implication; 4 is an instance of the S4.t -theorem $\Diamond_F\Box_PC \rightarrow C$; 5 is obtained from 4 by adding and distributing $\forall x$; and 7 follows from 1, 3, and 6. \square

Lemma A.2 *If A is a formula of \mathcal{L} , then $\text{Q}^\circ\text{S4.t} \vdash A^t \rightarrow \Box_FA^t$ and $\text{Q}^\circ\text{S4.t} \vdash \Diamond_PA^t \rightarrow A^t$.*

Proof. We only prove that $\text{Q}^\circ\text{S4.t} \vdash A^t \rightarrow \Box_FA^t$ since it implies that $\text{Q}^\circ\text{S4.t} \vdash \Diamond_PA^t \rightarrow A^t$. The proof is by induction on the complexity of A . If $A = \perp$, then $A^t = \perp$ and it is clear that $\text{Q}^\circ\text{S4.t} \vdash \perp \rightarrow \Box_F\perp$.

If A is either an atomic formula $P(x_1, \dots, x_n)$ or of the form $B \rightarrow C$ or $\forall xB$, then A^t is of the form \Box_FD . Therefore, the 4-axiom $\Box_FD \rightarrow \Box_F\Box_FD$ implies that in all these cases $\text{Q}^\circ\text{S4.t} \vdash A^t \rightarrow \Box_FA^t$.

If $A = \exists xB$, then $A^t = \Diamond_P\exists xB^t$. So $\Box_FA^t = \Box_F\Diamond_P\exists xB^t$ and $\text{Q}^\circ\text{S4.t} \vdash \Diamond_P\exists xB^t \rightarrow \Box_F\Diamond_P\exists xB^t$ because it is a substitution instance of the S4.t -theorem $\Diamond_PC \rightarrow \Box_F\Diamond_PC$. Finally, if $A = B \wedge C$ or $A = B \vee C$, then we have $A^t = B^t \wedge C^t$ or $A^t = B^t \vee C^t$. By inductive hypothesis, $\text{Q}^\circ\text{S4.t} \vdash B^t \rightarrow \Box_FB^t$ and $\text{Q}^\circ\text{S4.t} \vdash C^t \rightarrow \Box_FC^t$. Since $\text{Q}^\circ\text{S4.t} \vdash (\Box_FB^t \wedge \Box_FC^t) \rightarrow \Box_F(B^t \wedge C^t)$ and $\text{Q}^\circ\text{S4.t} \vdash (\Box_FB^t \vee \Box_FC^t) \rightarrow \Box_F(B^t \vee C^t)$, we obtain $\text{Q}^\circ\text{S4.t} \vdash (B^t \wedge C^t) \rightarrow \Box_F(B^t \wedge C^t)$ and $\text{Q}^\circ\text{S4.t} \vdash (B^t \vee C^t) \rightarrow \Box_F(B^t \vee C^t)$. \square

Lemma A.3 *The following are theorems of $\mathbb{Q}^\circ\text{S4.t}$:*

- (i) $\forall y(A(y/x) \rightarrow \exists xA)$.
- (ii) $\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall xB)$ if x is not free in A .
- (iii) $\forall x(A \rightarrow B) \rightarrow (\exists xA \rightarrow B)$ if x is not free in B .

Proof. Follows from [3, Lem. 1.3]. □

Lemma A.4 *For formulas A, B of \mathcal{L} , the following are theorems of $\mathbb{Q}^\circ\text{S4.t}$.*

- (i) $\Box_F(\Box_F\forall xA^t \rightarrow A^t)$ if x is not free in A .
- (ii) $\forall y\Box_F(\Box_F\forall xA^t \rightarrow A(y/x)^t)$.
- (iii) $\Box_F(A^t \rightarrow \Diamond_P\exists xA^t)$ if x is not free in A .
- (iv) $\forall y\Box_F(A(y/x)^t \rightarrow \Diamond_P\exists xA^t)$.
- (v) $\Box_F(\Box_F\forall x\Box_F(A^t \rightarrow B^t) \rightarrow \Box_F(A^t \rightarrow \Box_F\forall xB^t))$ if x is not free in A .
- (vi) $\Box_F(\Box_F\forall x\Box_F(A^t \rightarrow B^t) \rightarrow \Box_F(\Diamond_P\exists xA^t \rightarrow B^t))$ if x is not free in B .

Proof. Note that x is free in A iff it is free in A^t , and $A(y/x)^t = A^t(y/x)$.

(i). We have the proof

- 1. $\forall xA^t \rightarrow A^t$
- 2. $\Box_F\forall xA^t \rightarrow A^t$
- 3. $\Box_F(\Box_F\forall xA^t \rightarrow A^t)$

where 1 is an instance of NID because x is not free in A^t ; 2 is obtained from 1 by applying the T-axiom for \Box_F ; 3 is obtained from 2 by (N_F) .

(ii). We have the proof

- 1. $\forall y(\forall xA^t \rightarrow A^t(y/x))$
- 2. $\forall y(\Box_F\forall xA^t \rightarrow A^t(y/x))$
- 3. $\forall y\Box_F(\Box_F\forall xA^t \rightarrow A^t(y/x))$

where 1 is an instance of UI° ; 2 follows from 1 by applying the T-axiom for \Box_F inside $\forall y$; 3 is obtained from 2 by introducing \Box_F inside $\forall y$.

(iii). We have the proof

- 1. $A^t \rightarrow \exists xA^t$
- 2. $A^t \rightarrow \Diamond_P\exists xA^t$
- 3. $\Box_F(A^t \rightarrow \Diamond_P\exists xA^t)$

where 1 is an instance of $C \rightarrow \exists xC$, with x not free in C , which is equivalent to NID; 2 follows from 1 by the T-axiom for \Diamond_P ; 3 is obtained from 2 by (N_F) .

(iv). We have the proof

1. $\forall y(A^t(y/x) \rightarrow \exists xA^t)$
2. $\forall y(A^t(y/x) \rightarrow \diamond_P \exists xA^t)$
3. $\forall y \square_F(A^t(y/x) \rightarrow \diamond_P \exists xA^t)$

where 1 follows from Lemma A.3(i); 2 follows from 1 by applying the T-axiom for \diamond_P inside $\forall y$; 3 is obtained from 2 by introducing \square_F inside $\forall y$.

(v). We have the proof

1. $\forall x(A^t \rightarrow B^t) \rightarrow (A^t \rightarrow \forall xB^t)$
2. $\forall x \square_F(A^t \rightarrow B^t) \rightarrow (A^t \rightarrow \forall xB^t)$
3. $\square_F \forall x \square_F(A^t \rightarrow B^t) \rightarrow (\square_F A^t \rightarrow \square_F \forall xB^t)$
4. $\square_F \forall x \square_F(A^t \rightarrow B^t) \rightarrow (A^t \rightarrow \square_F \forall xB^t)$
5. $\square_F \forall x \square_F(A^t \rightarrow B^t) \rightarrow \square_F(A^t \rightarrow \square_F \forall xB^t)$
6. $\square_F(\square_F \forall x \square_F(A^t \rightarrow B^t) \rightarrow \square_F(A^t \rightarrow \square_F \forall xB^t))$

where 1 follows from Lemma A.3(ii); 2 follows from 1 by applying the T-axiom for \square_F ; 3 is obtained from 2 by adding and distributing \square_F ; 4 follows from 3 by Lemma A.2; 5 is obtained from 4 by adding and distributing \square_F and getting rid of one \square_F in the antecedent using the 4-axiom; 6 follows from 5 by (N_F) .

(vi). We have the proof

1. $\forall x(A^t \rightarrow B^t) \rightarrow (\exists xA^t \rightarrow B^t)$
2. $\forall x(A^t \rightarrow B^t) \rightarrow (\exists x \diamond_P A^t \rightarrow B^t)$
3. $\forall x \square_F(A^t \rightarrow B^t) \rightarrow (\exists x \diamond_P A^t \rightarrow B^t)$
4. $\forall x \square_F(A^t \rightarrow B^t) \rightarrow (\diamond_P \exists xA^t \rightarrow B^t)$
5. $\square_F \forall x \square_F(A^t \rightarrow B^t) \rightarrow \square_F(\diamond_P \exists xA^t \rightarrow B^t)$
6. $\square_F(\square_F \forall x \square_F(A^t \rightarrow B^t) \rightarrow \square_F(\diamond_P \exists xA^t \rightarrow B^t))$

where 1 follows from Lemma A.3(iii); 2 follows from 1 by Lemma A.2; 3 follows from 2 by applying the T-axiom for \square_F ; 4 follows from 3 and the fact that $Q^\circ S4.t \vdash \diamond_P \exists xA^t \rightarrow \exists x \diamond_P A^t$ because it is a consequence of BF_P ; 5 is obtained from 4 by adding and distributing \square_F ; 6 follows from 5 by (N_F) . \square

Lemma A.5 *If C is an instance of an axiom scheme of IQC and \mathbf{x} is the list of free variables in C , then $Q^\circ S4.t \vdash \forall \mathbf{x} C^t$.*

Proof. If C is an instance of a theorem of IPC, then it follows from the faithfulness of the Gödel translation in the propositional case that C^t is a theorem of $Q^\circ S4.t$ (since \square_F is an S4-modality). Applying (Gen) to each free variable of C^t then yields a proof of $\forall \mathbf{x} C^t$ in $Q^\circ S4.t$. Translations of the axiom schemes of Definition 2.1 give:

$$\begin{aligned} (\forall xA \rightarrow A(y/x))^t &= \square_F(\square_F \forall xA^t \rightarrow A(y/x)^t) \\ (A(y/x) \rightarrow \exists xA)^t &= \square_F(A(y/x)^t \rightarrow \diamond_P \exists xA^t) \end{aligned}$$

$$\begin{aligned}
& (\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall xB))^t \\
& \quad = \Box_F(\Box_F \forall x \Box_F(A^t \rightarrow B^t) \rightarrow \Box_F(A^t \rightarrow \Box_F \forall x B^t)) \\
& (\forall x(A \rightarrow B) \rightarrow (\exists xA \rightarrow B))^t \\
& \quad = \Box_F(\Box_F \forall x \Box_F(A^t \rightarrow B^t) \rightarrow \Box_F(\Diamond_P \exists x A^t \rightarrow B^t))
\end{aligned}$$

If C is an instance of one of these axiom schemes, then we obtain a proof of $\forall \mathbf{x} C^t$ in $\mathsf{Q}^\circ\mathsf{S4.t}$ by Lemma A.4 and by applying (Gen) to the free variables of C . More precisely, for the first axiom we use (i) of Lemma A.4 when x is not free in A and (ii) when x is free in A . Similarly, for the second axiom we use (iii) or (iv) of Lemma A.4. Finally, for the third axiom we use (v) and for the fourth axiom we use (vi) of Lemma A.4. \square

Lemma A.6 *Let A, B be formulas of \mathcal{L} , \mathbf{x} the list of variables free in $A \rightarrow B$, \mathbf{y} the list of variables free in A , and \mathbf{z} the list of variables free in B . If $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{x}(A \rightarrow B)^t$ and $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{y}A^t$, then $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{z}B^t$.*

Proof. Let \mathbf{u} be the list of variables free in A but not in B , \mathbf{v} the list of variables free in B but not in A , and \mathbf{w} the list of variables free in both A and B . We then have that \mathbf{x} is the union of \mathbf{u} , \mathbf{v} , and \mathbf{w} ; \mathbf{y} is the union of \mathbf{u} and \mathbf{w} ; and \mathbf{z} is the union of \mathbf{v} and \mathbf{w} . Thus, we want to show that if $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{u} \forall \mathbf{v} \forall \mathbf{w}(A \rightarrow B)^t$ and $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{u} \forall \mathbf{w}A^t$, then $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{v} \forall \mathbf{w}B^t$. We have the proof

1. $\forall \mathbf{u} \forall \mathbf{v} \forall \mathbf{w} \Box_F(A^t \rightarrow B^t)$
2. $\forall \mathbf{u} \forall \mathbf{w} \forall \mathbf{v} \Box_F(A^t \rightarrow B^t)$
3. $\forall \mathbf{u} \forall \mathbf{w} \forall \mathbf{v} (\Box_F A^t \rightarrow \Box_F B^t)$
4. $\forall \mathbf{u} \forall \mathbf{w} (\Box_F A^t \rightarrow \forall \mathbf{v} \Box_F B^t)$
5. $\forall \mathbf{u} \forall \mathbf{w} \Box_F A^t \rightarrow \forall \mathbf{u} \forall \mathbf{w} \forall \mathbf{v} \Box_F B^t$
6. $\forall \mathbf{u} \forall \mathbf{w} A^t$
7. $\forall \mathbf{u} \forall \mathbf{w} \Box_F A^t$
8. $\forall \mathbf{u} \forall \mathbf{w} \forall \mathbf{v} \Box_F B^t$
9. $\forall \mathbf{u} \forall \mathbf{w} \forall \mathbf{v} B^t$
10. $\forall \mathbf{w} \forall \mathbf{v} B^t$
11. $\forall \mathbf{v} \forall \mathbf{w} B^t$

where 1 and 6 are assumptions; 2 and 11 follow from 1 and 10 by switching the order of quantification; 3 is obtained from 2 by distributing \Box_F inside the universal quantifiers; 4 follows from Lemma A.3(ii) because all the variables in \mathbf{v} are not free in $\Box_F A^t$; 5 is obtained by distributing the universal quantifiers; 7 follows from 6 by introducing \Box_F inside the quantifiers; 8 is obtained by (MP) from 5 and 7; 9 follows from 8 by the T-axiom for \Box_F ; 10 follows from 9 by NID because no variable in \mathbf{u} is free in B^t . \square

Lemma A.7 *Let A be a formula of \mathcal{L} , x a variable, \mathbf{y} the list of variables free in A , and \mathbf{z} the list of variables free in $\forall xA$. If $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{y}A^t$, then $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{z}(\forall xA)^t$.*

Proof. If x is in \mathbf{y} , then without loss of generality we may assume that \mathbf{y} is \mathbf{z} concatenated with x . Therefore, by assumption we have $\mathbf{Q}^\circ\mathbf{S4.t} \vdash \forall \mathbf{z} \forall x A^t$. If x is not in \mathbf{y} , then $\mathbf{y} = \mathbf{z}$. Thus, by (Gen) for x and by switching the order of quantifiers, we again obtain $\mathbf{Q}^\circ\mathbf{S4.t} \vdash \forall \mathbf{z} \forall x A^t$. We can then introduce \square_F inside the quantifiers to obtain $\mathbf{Q}^\circ\mathbf{S4.t} \vdash \forall \mathbf{z} \square_F \forall x A^t$ which means $\mathbf{Q}^\circ\mathbf{S4.t} \vdash \forall \mathbf{z} (\forall x A)^t$. \square

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