A Remark on the Superintuitionistic Predicate Logic of Kripke Frames of Finite Height with Constant Domains: A Simpler Kripke Complete Logic That Is Not Strongly Complete

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Abstract

We show that the superintuitionistic predicate logic characterized by all Kripke frames of finite height with constant domains is not strongly Kripke complete (as well as some its extensions). This gives new examples of Kripke complete logics that are not strongly complete, cf. Problem 1 in [5]; the previous examples of such logics found by Takano were Π^1_1-hard, while ours are Π_2-arithmetical.

Keywords: Superintuitionistic predicate logics, Kripke semantics, Kripke completeness, strong Kripke completeness.

H. Ono (the talk at L.E.J. Brouwer Centenary Symposium held in 1981, cf. [5, Problem 1 (P34)]) asked if every Kripke complete intermediate predicate logic is strongly Kripke complete. M. Takano found a counterexample, mentioned by Ono in [5]: namely, *the logic of any Kripke frame with a denumerable constant domain, whose set of worlds is an infinite ordinal, is not strongly Kripke complete.* Note that all these logics are Π^1_1-hard (cf. [9,11,12]).

Here we consider the intermediate predicate logics $\mathbf{LP}_\infty$ and $\mathbf{L}^c\mathbf{P}_\infty$ characterized by all Kripke frames of finite height (with expanding and with constant domains respectively). We show that the logic $\mathbf{L}^c\mathbf{P}_\infty$ (with constant domains) is not strongly Kripke complete; moreover, a slightly weakened version of strong completeness fails as well (for this logic and for many its extensions). The similar question for the logic $\mathbf{LP}_\infty$ (with expanding domains) remains open. Note that the logics $\mathbf{LP}_\infty$ and $\mathbf{L}^c\mathbf{P}_\infty$ are not recursively axiomatizable (see [11]); on the other hand, they are obviously Π_2-arithmetical. Namely, $\mathbf{L}^c\mathbf{P}_\infty = \bigcap_n \mathbf{L}^c\mathbf{P}_n$, where $\mathbf{L}^c\mathbf{P}_n$ is a finitely axiomatizable logic of Kripke frames of height n (with constant domains); and similarly for the case with expanding domains.

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We do not know *if every recursively axiomatizable (or at least every finitely axiomatizable) Kripke complete predicate logic is strongly Kripke complete.*

The main result of the paper was announced in [13].

Remark 0. By the way, H. Ono formulated a similar question for intermediate propositional logics [5, Problem 1′ (P35)]. T. Shimura [8] found a family of propositional counterexamples; namely, he obtained the following result:

(**Sh**): *Intuitionistic logic is the only strongly complete intermediate propositional logic weaker than* $\mathbf{GJ}_2$ (Gabbay – de Jongh's logic of finite binary trees [1]).

This implies the following consequence for the predicate case: [2]

(**Sh**)′ : *Every intermediate predicate logic, the propositional fragment of which is included in* $\mathbf{GJ}_2$ *and is non-intuitionistic* (i.e., it is not equal to intuitionistic logic), *is not strongly Kripke complete.*

In particular, we can see that for any $n \geq 2$, the predicate logics, characterized by all Kripke frames (with constant and with expanding domains) over finite n-ary trees, are not strongly Kripke complete (note that these logics are Π_2-arithmetical, and they are not RE by [10]). On the other hand, the claim (**Sh**), as well as any similar result for the propositional case, obviously does not give anything for predicate logics with the intuitionistic propositional fragment (like $\mathbf{L}^c\mathbf{P}_\infty$, $\mathbf{LP}_\infty$, and many other logics considered in the present paper).

By the way, V. Shehtman [6] showed that every Kripke complete intermediate propositional logic is strongly Kripke complete in the topological semantics. We do not know if this result transfers to predicate logics.

Section 1. Preliminary notions

We consider superintuitionistic predicate logics without equality and function symbols (called in this paper *predicate logics*, or sometimes even *logics*, for short [3]). These are defined as extensions of intuitionistic predicate logic **QH** closed under modus ponens, generalization, and substitution of arbitrary formulas for atomic ones (cf. e.g. [2, Definition 2.6.3]; the book contains the basic notions in the field). For these logics we use the standard predicate Kripke semantics. Let us recall the corresponding definitions.

1.1 A *predicate Kripke frame with expanding domains* (or a *Kripke frame*, or even a *frame*, for short) is a pair (M, U) with M a non-empty partially ordered set (poset) and U a domain map defined on M such that, for any $u, v \in M$:

$(i)\ U(u) \neq \varnothing$, and $(ii)\ u \leq v \Rightarrow U(u) \subseteq U(v)$.

We say that (M, U) is a Kripke frame *over* the poset M. If U is a constant mapping on M such that $U(u) = X$ for all $u \in M$, then we write $(M, \lambda u.X)$ for (M, U) and call it *a Kripke frame with a constant domain.* [4]

[2] In [8] this straightforward corollary was not mentioned and Ono's Problem 1 (P34) was not addressed.

[3] in essence, we do not consider other sorts of logics

[4] The notation $(M, \lambda u.X)$ was introduced in [10], while in [11] we used the notation $(M; X)$.

The notions of a valuation and of validity of a predicate formula on a Kripke frame are defined in a usual way (cf. e.g. [15]). Namely, a *valuation* on (M, U) is a forcing relation $u \vDash A$ between points $u \in M$ and formulas A (with parameters replaced by elements of $U(u)$), satisfying *monotonicity* : $u \leq v$, $u \vDash A \Rightarrow v \vDash A$ and the following inductive conditions:

$$u \vDash (B \& C) \Leftrightarrow (u \vDash B) \,\&\, (u \vDash C); \quad u \vDash (B \vee C) \Leftrightarrow (u \vDash B) \vee (u \vDash C);$$
$$u \vDash (B \supset C) \Leftrightarrow \forall v \geq u \,[\,(v \vDash B) \Rightarrow (v \vDash C)\,]; \qquad u \nvDash \bot;$$
$$u \vDash \forall x B(x) \Leftrightarrow \forall v \geq u \, \forall c \in U(v) \,[\, v \vDash B(c)\,];$$
$$u \vDash \exists x B(x) \Leftrightarrow \exists c \in U(u) \,[\, u \vDash B(c)\,].$$

As usual, to obtain a valuation, it is sufficient to know $u \vDash A$ only for atomic A, and by induction, the monotonicity condition for atomic A implies the monotonicity for arbitrary (non-atomic) formulas A.

A Kripke frame with a valuation is called a *Kripke model.*

A predicate formula $A(x_1, \dots, x_n)$ is said to be *true* (under a valuation $\vDash$ on a frame (M, U)) if $u \vDash A(a_1, \dots, a_n)$ for any $u \in M$ and $a_1, \dots, a_n \in U(u)$. A formula A is *valid* on a Kripke frame (M, U) if it is true under any valuation on (M, U). The *predicate logic* $\mathbf{L}(M, U)$ *of a Kripke frame* (M, U) is the set of all formulas valid on (M, U). It is well known that this set is indeed a superintuituionistic logic.

1.2 The *predicate logic of a class* $\mathbf{Z}$ *of Kripke frames* is

$$\mathbf{L}[\mathbf{Z}] = \bigcap (\, \mathbf{L}(M, U) : (M, U) \in \mathbf{Z}\,).$$

A predicate logic $\mathbf{L}$ is *Kripke complete* if $\mathbf{L} = \mathbf{L}[\mathbf{Z}]$ for some class $\mathbf{Z}$ of Kripke frames. The *Kripke completion* of a logic $\mathbf{L}$ is $\mathbf{L}^+ = \bigcap (\mathbf{L}(M, U) : \mathbf{L} \subseteq \mathbf{L}(M, U)\,)$, the smallest (w.r.t. the inclusion) Kripke complete extension of $\mathbf{L}$.

The *predicate logic of a poset* M is $\mathbf{L}M = \bigcap_U \mathbf{L}(M, U)$ and the *predicate logic of a class* $\mathbf{Y}$ *of posets* is $\mathbf{LY} = \bigcap (\, \mathbf{L}M \,:\, M \in \mathbf{Y}\,)$. Analogously, we define the *predicate logic with constant domains of a class* $\mathbf{Y}$:

$$\mathbf{L}^c\mathbf{Y} = \bigcap (\, \mathbf{L}(M, \lambda u.X) \,:\, M \in \mathbf{Y},\, X \neq \varnothing\,).$$

A poset M with the least element 0_M is called *rooted.* A *cone* in a poset M is $M^u = \{v \in M \mid u \leq v\}$ (for $u \in M$). The *cone* (M^u, U) of a Kripke frame (M, U) is its restriction to M^u.

It is well known that (I) $\mathbf{L}(M, U) = \bigcap (\, \mathbf{L}(M^u, U) : u \in M\,)$,

hence (II) $\mathbf{LY} = \mathbf{L}\{M^u : M \in \mathbf{Y},\, u \in M\}$ for a class $\mathbf{Y}$ of posets (and similarly with $\mathbf{L}^c\mathbf{Y}$). This means that rooted Kripke frames (and rooted posets) are sufficient for the Kripke semantics of superintuitionistic logics.

1.3 Let us consider the *constant domain principle*:

$$D = \forall x (Q(x) \vee p) \supset \forall x Q(x) \vee p,$$

where p is a propositional symbol and Q is a unary predicate symbol. It is well known that D is valid on any Kripke frame with a constant domain (i.e., on any frame of the form $(M, \lambda u.X)$ with $X \neq \varnothing$); hence $D \in \mathbf{L}^c M$ for any poset M and $D \in \mathbf{L}^c \mathbf{Y}$ for any class $\mathbf{Y}$ of posets. Moreover, for a rooted poset M the following equivalence holds:

$$D \in \mathbf{L}(M,U) \quad \textit{iff} \quad \textit{a frame } (M,U) \textit{ has a constant domain.} \qquad (\delta)$$

Note that for non-rooted M, this equivalence does not hold in general; e.g., D is valid on a disjoint union of two frames with different constant domains.

1.4 Recall that the *height* $h[M]$ of a poset M is the supremum of cardinalities of chains (i.e., linearly ordered subsets) in M. Similarly, the *width* $w[M]$ of a rooted poset M is the supremum of cardinalities of antichains (i.e., sets of pairwise incomparable elements) in M. The width of an arbitrary poset is $w[M] = \sup(w[M^u] : u \in M)$).

Let $\mathbf{P}_n$ be the class of all posets of height $h[M] \leq n$ (for $n \in \omega,\ n > 0$) and let $\mathbf{P}_\infty = \bigcup\limits_n \mathbf{P}_n$ be the class of posets of finite height. Analogously, one can introduce the class $\mathbf{W}_n$ of all posets of width $w[M] \leq n$ (and $\mathbf{W}_\infty = \bigcup\limits_n \mathbf{W}_n$, the class of posets of finite width).

Let S_n be an n-element chain, $n > 0$; clearly, its height is n and its width is 1. Denote $\mathbf{S}_\infty = \{S_n : n \in \omega,\ n > 0\}$. Then $\mathbf{L}\mathbf{S}_\infty = \bigcap\limits_n \mathbf{L} S_n$ and $\mathbf{L}^c \mathbf{S}_\infty = \bigcap\limits_n \mathbf{L}^c S_n$; similarly $\mathbf{L}\mathbf{P}_\infty = \bigcap\limits_n \mathbf{L}\mathbf{P}_n$ and $\mathbf{L}^c \mathbf{P}_\infty = \bigcap\limits_n \mathbf{L}^c \mathbf{P}_n$, etc. Finally, let **Fin** be the class of all finite posets. Obviously $\mathbf{S}_\infty = \mathbf{P}_\infty \cap \mathbf{W}_1$ and $\mathbf{Fin} = \mathbf{P}_\infty \cap \mathbf{W}_\infty$; so $\mathbf{S}_\infty \subset \mathbf{Fin} \subset \mathbf{P}_\infty$.

It can be easily shown (applying [3, Theorem 3.4]) that

$$\mathbf{L}M \subseteq \mathbf{L}S_n \textit{ and } \mathbf{L}^c M \subseteq \mathbf{L}^c S_n \textit{ for any poset } M \textit{ of height } h[M] \geq n, \qquad (\sigma)$$

because the n-element chain S_n is a p-morphic image of any poset M of height $h[M] \geq n$ (note that p-morphisms were called embeddings in [3, Section 3]).

Actually moreover: $\mathbf{L}M \subseteq \mathbf{L}S_n$ *iff* $\mathbf{L}^c M \subseteq \mathbf{L}^c S_n$ *iff* $h[M] \geq n$, because $P_{n-1} \in \mathbf{L}M \backslash \mathbf{L}^c S_n$ if $h[M] < n$ (the formulas P_n of height n are defined in Section 2).

Section 2. Main result

2.1 Let $\mathbf{L}$ be a logic and Γ, Δ be two sets of sentences. A pair (Γ, Δ) is called $\mathbf{L}$-*inconsistent* if $\mathbf{L} \vdash (\&\Gamma_0 \supset \bigvee \Delta_0)$ for some finite subsets $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$. A pair (Γ, Δ) is *satisfiable* in a Kripke frame (M, U) if there exists a valuation in (M, U) and a world $u \in M$ such that $u \vDash A$ for all formulas $A \in \Gamma$ and $u \nvDash B$ for all formulas $B \in \Delta$. We say that a predicate logic $\mathbf{L}$ is *strongly Kripke complete without parameters* if every $\mathbf{L}$-consistent pair (Γ, Δ) of sets of sentences is satisfiable in a Kripke frame validating $\mathbf{L}$. The usual notion of *strong Kripke completeness* (given in [5]) is slightly stronger: namely, formulas with parameters (not necessarily sentences) are allowed in Γ and Δ; naturally, these parameters would be evaluated by individuals taken from the corresponding domain $U(u)$.

Lemma 1 (**Main Lemma**) *There exists a predicate sentence* $A^*_{\mathbf{P}_\infty}$ (or A^* for short) *such that for every Kripke frame* (M,U)*:* $A^*_{\mathbf{P}_\infty} \in \mathbf{L}(M,U)$ *iff*

$$\textit{for all } u \textit{ in } M \; [\; \textit{the height } h[M^u] \textit{ is finite, or the domain } U(u) \textit{ is finite} \;].$$

Hence for rooted Kripke frames with constant domains we have:[5]

$$A^*_{\mathbf{P}_\infty} \in \mathbf{L}(M,\lambda u.X) \;\; \textit{iff} \;\; [\; \textit{the height } h[M] \textit{ is finite, or the domain } X \textit{ is finite} \;].$$

Thus, clearly $\quad A^*_{\mathbf{P}_\infty} \in \mathbf{LP}_\infty, \quad$ so $\quad A^*_{\mathbf{P}_\infty} \& D \in \mathbf{L}^c\mathbf{P}_\infty$.

We present the proof of this lemma in Section 4.

2.2 Now we apply Main Lemma to show that strong Kripke completeness fails for $\mathbf{L}^c\mathbf{P}_\infty$; more precisely, we prove:

Theorem 1 *The logic* $\mathbf{L}^c\mathbf{P}_\infty$ *of frames of finite height with constant domains is not strongly Kripke complete without parameters.*

Take the following propositional formulas of finite heights:

$$P_0 = \bot \text{ and } P_{n+1} = p_n \vee (p_n \supset P_n) \text{ for } n \in \omega,$$

where $p_0, \dots, p_n, \dots$ are different propositional symbols. It is well known that: $P_n \in \mathbf{L}(M,U)$ iff $h[M] \leq n$. Hence $P_n \in \mathbf{LP}_n$ and $(P_n \& D) \in \mathbf{L}^c\mathbf{P}_n$ for any $n > 0$. By the way, note that $\mathbf{L}^c\mathbf{P}_n = [\mathbf{QH} + P_n \& D]$ (cf. [4, Theorem 3.3]), while $\mathbf{LP}_n \neq [\mathbf{QH} + P_n]$ (i.e., the logics $[\mathbf{QH} + P_n]$ are Kripke incomplete) for $n \geq 2$ (see [4, Theorem 3.2]); by definition, $[\mathbf{QH} + P_1]$ is classical logic.[6]

Also take the sentences

$$C_m = \forall x_1, \dots, \forall x_m \,[\, \mathop{\&}_i Q_i(x_i) \supset \bigvee_{i<j} Q_i(x_j) \,]$$

for $m > 1$. Clearly, $\quad C_m \in \mathbf{L}(M,\lambda u.X) \;$ iff $\;$ card$(X) < m$.

Take the set $\Delta^* = \{P_n : n > 0\} \cup \{C_m : m > 1\}$. Then the pair $(\varnothing, \Delta^*)$ is $\mathbf{L}^c\mathbf{P}_\infty$-consistent (and moreover, it is $\mathbf{L}^c\mathbf{S}_\infty$-consistent). Indeed, any finite $\Delta_0 \subseteq \Delta^*$ is included in $\{P_n : n < n_0\} \cup \{C_m : m \leq m_0\}$ for some n_0, m_0; then the corresponding disjunction $\bigvee \Delta_0$ is falsified in every Kripke frame of height n_0 (in particular, in S_{n_0}) with an m_0-element constant domain.

On the other hand, the subsequent claim shows that the pair $(\varnothing, \Delta^*)$ is not satisfiable in $\mathbf{L}^c\mathbf{P}_\infty$-frames; hence the logic $\mathbf{L}^c\mathbf{P}_\infty$ is not strongly Kripke complete without parameters.

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[5] For non-rooted M, this equivalence does not hold in general; e.g., $A^*_{\mathbf{P}_\infty}$ is valid on a disjoint union M of all finite chains S_n (with an infinite constant domain), while $h[M]$ is infinite.

[6] Note that the logics $\mathbf{LP}_n$ are finitely axiomatizable for all $n > 0$; their axioms are P_n^+, which are essentially predicate formulas similar to P_n (see [16]).

Claim *Let* (M, U) *be a rooted Kripke frame validating* $\mathbf{L}^c\mathbf{P}_\infty$.
Then $\mathbf{L}(M, U) \cap \Delta^* \neq \varnothing$.

Proof. Clearly, $D \in \mathbf{L}^c\mathbf{P}_\infty \subseteq \mathbf{L}(M, U)$, and so (M, U) has a constant domain (due to (δ), see in Section 1). Now, if $\mathbf{L}(M, U) \cap \Delta^* = \varnothing$, i.e., all P_n and C_m are falsified in (M, U), then its domain is infinite and the height $h(M)$ is infinite as well. Thus $A^*_{\mathbf{P}_\infty} \notin \mathbf{L}(M, U)$ and so $\mathbf{L}^c\mathbf{P}_\infty \nsubseteq \mathbf{L}(M, U)$ (since $A^*_{\mathbf{P}_\infty} \in \mathbf{L}^c\mathbf{P}_\infty$). □

Actually, our argument gives a more general result:

Theorem 2 *Let* $\mathbf{L}$ *be a predicate logic such that* $\mathbf{L}^c\mathbf{P}_\infty \subseteq \mathbf{L} \subseteq \mathbf{L}^c\mathbf{S}_\infty$. *Then* $\mathbf{L}$ *is not strongly Kripke-complete without parameters.*

Indeed, if $\mathbf{L} \subseteq \mathbf{L}^c\mathbf{S}_\infty$, then the pair $(\varnothing, \Delta^*)$ is $\mathbf{L}$-consistent. And if $\mathbf{L}^c\mathbf{P}_\infty \subseteq \mathbf{L}$, then $(\varnothing, \Delta^*)$ is not satisfiable in $\mathbf{L}$-frames. □

Corollary 1 *Let* $\mathbf{Y} \subseteq \mathbf{P}_\infty$ *be a class of posets of finite height such that* $\forall n \in \omega(\mathbf{Y} \nsubseteq \mathbf{P}_n)$ (*i.e.,* $\mathbf{Y}$ *contains posets of arbitrarily large heights*). *Then the Kripke complete logic* $\mathbf{L}^c\mathbf{Y}$ *is not strongly Kripke complete* (*even without parameters*).

Indeed, if $\mathbf{Y} \nsubseteq \mathbf{P}_n$, then $\mathbf{L}^c\mathbf{Y} \subseteq \mathbf{L}^c\mathbf{S}_n$, due to (σ) (see the end of Section 1). □

Therefore, we conclude that the following Kripke complete logics (and many other ones) are not strongly Kripke complete (without parameters): [7]

$$\mathbf{L}^c\mathbf{P}_\infty, \mathbf{L}^c\mathbf{Fin}, \mathbf{L}^c\mathbf{S}_\infty = \mathbf{L}^c(\mathbf{P}_\infty \cap \mathbf{W}_1), \mathbf{L}^c(\mathbf{P}_\infty \cap \mathbf{W}_m) = \mathbf{L}^c(\mathbf{Fin} \cap \mathbf{W}_m),$$
$$\mathbf{L}^c(\mathbf{P}_n \cup (\mathbf{P}_\infty \cap \mathbf{W}_m)) = \mathbf{L}^c\mathbf{P}_n \cap \mathbf{L}^c(\mathbf{Fin} \cap \mathbf{W}_m) \text{ for every } m, n \in (\omega \setminus \{0\}). \quad (\lambda)$$

Note that all logics $\mathbf{L}$ mentioned in Theorem 2 (in particular, all logics listed in (λ)) are not recursively axiomatizable, see [11, Theorem 1.2]. On the other hand, all logics listed in (λ) are Π_2-arithmetical, because the logics $\mathbf{L}^c\mathbf{P}_n$ for $n < \omega$ are finitely axiomatizable (see in Section 1) and all logics $\mathbf{L}^cM$ for finite posets M are recursively axiomatizable (in a uniform way), see e.g. [10]. [8]

Section 3. A short discussion and open questions

3.1 Main Lemma shows that, for the Kripke semantics with constant domains, the formula $A^*_{\mathbf{P}_\infty}$ (or, more precisely, $A^*_{\mathbf{P}_\infty} \& D$) describes the finiteness of height, up to a minor additional exception involving (arbitrary) Kripke frames with finite constant domain. Now we will explain why this addition is inevitable.

Let $\mathbf{Kr}^c_m = \{(M, \lambda u.X) \mid M \neq \varnothing, \mathrm{card}(X) = m\}$ be the class of Kripke frames with m-element constant domain (for $m \in \omega, m > 0$). Now, let

$$\mathbf{Kr}^c_\infty = \bigcup_m \mathbf{Kr}^c_m = \{\, (M, \lambda u.X) \mid M \neq \varnothing, X \text{ is finite } \}$$

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[7] Note that all these examples are not covered by the claim $(\mathbf{Sh})'$ (see Remark 0 at the beginning of the paper), because the propositional fragments of these logics are either intuitionistic or not included in $\mathbf{GJ}_2$.

[8] Moreover, they are finitely axiomatizable by [7, Theorem 3.7] (cf. our Proposition 2 in Section 3).

be the class of Kripke frames with finite constant domains.

Similarly, we introduce the classes of frames with finite M:

$\mathbf{Fin}^c_m = \{(M, \lambda u.X) \mid M \text{ is finite}, \mathrm{card}(X) = m\} = \{(M, \lambda u.X) \in \mathbf{Kr}^c_m \mid M \in \mathbf{Fin}\}$

and $\mathbf{Fin}^c_\infty = \bigcup\limits_m \mathbf{Fin}^c_m = \{(M, \lambda u.X) \mid M \text{ and } X \text{ are finite }\} =$

$= \{(M, \lambda u.X) \in \mathbf{Kr}^c_\infty \mid M \in \mathbf{Fin}\}$.

Finally, we have analogous classes of frames of finite height:

$\mathbf{P}^c_{\infty,m} = \{(M, \lambda u.X) \mid M \in \mathbf{P}_\infty, \mathrm{card}(X) = m\} = \{(M, \lambda u.X) \in \mathbf{Kr}^c_m \mid M \in \mathbf{P}_\infty\}$

and $\mathbf{P}^c_{\infty,\infty} = \bigcup\limits_m \mathbf{P}^c_{\infty,m} = \{ (M, \lambda u.X) \mid h[M] \text{ and } X \text{ are finite } \} =$

$= \{(M, \lambda u.X) \in \mathbf{Kr}^c_\infty \mid M \in \mathbf{P}_\infty\}$.

Clearly, $\mathbf{Fin}^c_m \subset \mathbf{P}^c_{\infty,m} \subset \mathbf{Kr}^c_m$ for any m, and so $\mathbf{Fin}^c_\infty \subset \mathbf{P}^c_{\infty,\infty} \subset \mathbf{Kr}^c_\infty$. Also $\mathbf{L}[\mathbf{Kr}^c_\infty] = \bigcap\limits_m \mathbf{L}[\mathbf{Kr}^c_m]$ and similarly with $\mathbf{L}[\mathbf{Fin}^c_\infty]$, $\mathbf{L}[\mathbf{P}^c_{\infty,\infty}]$.

Lemma 2 $\mathbf{L}[\mathbf{Kr}^c_m] = \mathbf{L}[\mathbf{Fin}^c_m]$,
and so $\mathbf{L}[\mathbf{Kr}^c_m] = \mathbf{L}[\mathbf{P}^c_{\infty,m}] = \mathbf{L}[\mathbf{Fin}^c_m]$ *for every* $m \in \omega, m > 0$.

Therefore, $\mathbf{L}[\mathbf{Kr}^c_\infty] = \mathbf{L}[\mathbf{P}^c_{\infty,\infty}] = \mathbf{L}[\mathbf{Fin}^c_\infty]$.

In other words, for the Kripke semantics with finite constant domains, we have:

every finitely valid formula is generally valid, i.e.,

every formula valid on all finite posets is valid on all (non-empty) posets M.

Hence we obtain:

Corollary 2 $\mathbf{L}^c\mathbf{Fin} \subset \mathbf{L}[\mathbf{Kr}^c_\infty]$, *and thus* $\mathbf{L}^c\mathbf{P}_\infty \subset \mathbf{L}^c\mathbf{Fin} \subset \mathbf{L}[\mathbf{Kr}^c_\infty]$.

This means that any formula valid (in the semantics with constant domains) on all frames (with arbitrary domains) over finite posets M (in particular, any formula valid on all frames of finite height) is necessarily valid on all frames with finite domains (with arbitrary posets M). That is why our sentence $A^*_{\mathbf{P}_\infty}$ (presented in Lemma 1), which is valid on all frames of finite height, is inevitably valid on all frames with finite constant domains (with arbitrary M).

Remark 1 By the way, the inclusion $\mathbf{L}^c\mathbf{Fin} \subset \mathbf{L}[\mathbf{Kr}^c_\infty]$ is definitely proper. Indeed, let A be a well-known formula (with one binary predicate symbol R) that is classically valid on all finite domains and is not valid on infinite domains (e.g. $A = \neg A_0$ for the formula A_0 defined at the beginning of Section 4). Put $A' = [\forall x, y(R(x,y) \vee \neg R(x,y)) \supset A]$. Then A' belongs to $\mathbf{L}[\mathbf{Kr}^c_\infty] \backslash \mathbf{QC}$, where $\mathbf{QC}$ is classical predicate logic; so all the more it belongs to $\mathbf{L}[\mathbf{Kr}^c_\infty] \backslash \mathbf{L}^c\mathbf{Fin}$.

Remark 2 The inclusion $\mathbf{L}^c\mathbf{P}_\infty \subset \mathbf{L}^c\mathbf{Fin}$ is proper as well. Indeed, there exists a formula $A^*_{\mathbf{Fin}}$ (introduced in [10] and denoted by F' there) such that (cf. our Lemma 1):

$$A^*_{\mathbf{Fin}} \in \mathbf{L}(M, U) \textit{ iff } \forall u \in M\, [\, M^u \textit{ or } U(u) \textit{ is finite}]. \quad (\varphi)$$

Then clearly $A^*_{\mathbf{Fin}} \in \mathbf{LFin} \backslash \mathbf{L}^c\mathbf{P}_\infty$, and so $A^*_{\mathbf{Fin}} \in \mathbf{L}^c\mathbf{Fin} \backslash \mathbf{L}^c\mathbf{P}_\infty$. Moreover, it is easily seen that $\mathbf{LFin} \not\subseteq \mathbf{L}^c\mathbf{Y}$ for any class $\mathbf{Y}$ of rooted posets that contains an infinite poset, and so $\mathbf{LY} \subset \mathbf{LFin}$ and $\mathbf{L}^c\mathbf{Y} \subset \mathbf{L}^c\mathbf{Fin}$ for any class $\mathbf{Y} \supset \mathbf{Fin}$ that contains a poset with an infinite cone. The similar statement holds for $\mathbf{LP}_\infty$ as well, due to our formula $A^*_{\mathbf{P}_\infty}$.

Now we prove Lemma 2.

Proof. Let $X_m = \{1, \ldots, m\}$ be an m-element domain. To every k-ary predicate symbol P and all $j_1, \ldots, j_k \in X_m$, we assign a unique propositional symbol $\bar{P}^{(j_1,\ldots,j_k)}$. For a predicate formula $A(i_1, \ldots, i_n)$ with parameters replaced by elements of X_m one can easily construct a propositional formula $\bar{A}_{(m)}^{(i_1,\ldots,i_n)}$, which simulates A in a natural way; namely, we replace predicate atoms $P(j_1, \ldots, j_k)$ with propositional atoms $\bar{P}^{(j_1,\ldots,j_k)}$ and replace quantifiers $\forall$ and $\exists$ with the conjunction and disjunction over all elements of X_m. One can easily show (by induction) that the truth of $A(i_1, \ldots, i_n)$ in $(M, \lambda u.X_m)$ (at a point $v \in M$) is equivalent to the propositional truth of $\bar{A}_{(m)}^{(i_1,\ldots,i_n)}$ in M at the same point v (under the corresponding valuations of symbols P in $(M, \lambda u.X_m)$ and $\bar{P}^{(i_1,\ldots,i_k)}$ in M). Therefore, we conclude that $A \in \mathbf{L}(M, \lambda u.X_m)$ iff $\bar{A}_{(m)} \in \mathbf{L}(M)$ for any sentence A and a poset (i.e., a propositional Kripke frame) M.

Finally, we use the following well-known fact: any propositional formula valid on all finite posets M is intuitionistically provable, and so it is valid on all (non-empty) M as well. □

3.2 Hence we obtain:

Proposition 1 *Let (M,U) be a rooted Kripke frame. Then the following conditions are equivalent*:

(1) $\mathbf{L}^c\mathbf{P}_\infty \subseteq \mathbf{L}(M,U)$;

(2) $(\mathbf{LP}_\infty + D) \subseteq \mathbf{L}(M,U)$;

(3) $(A^*_{\mathbf{P}_\infty} \& D) \in \mathbf{L}(M,U)$;

(4) (M,U) *is a Kripke frame with constant domain X (i.e., it is $(M, \lambda u.X)$) and [the height $h(M)$ or the domain X is finite].*

Proof. The implication (1) ⇒ (2) is obvious (since $D \in \mathbf{L}^c\mathbf{P}_\infty$, see in Section 1). The implications (2) ⇒ (3) ⇒ (4) readily follow from Lemma 1 and (δ) (again see in Section 1). And the implication (4) ⇒ (1) follows from Corollary 2. □

Therefore, $\mathbf{L}^c\mathbf{P}_\infty$ *is the Kripke completion of* $(\mathbf{QH} + A^*_{\mathbf{P}_\infty} \& D)$.

Note that the logic $(\mathbf{QH} + A^*_{\mathbf{P}_\infty} \& D)$ is Kripke incomplete, because its Kripke completion $\mathbf{L}^c\mathbf{P}_\infty$ is not recursively axiomatizable, as it was mentioned in Section 2 (cf. [11]). The logic $(\mathbf{QH} + A^*_{\mathbf{P}_\infty})$ is Kripke incomplete as well; its Kripke completion described by Lemma 1 is not RE (see [11, Theorem 2.2]). [9]

Now let us state two related open questions:

Question 1 Is $(\mathbf{LP}_\infty + D)$ equal to $\mathbf{L}^c\mathbf{P}_\infty$?

In other words, is the logic $(\mathbf{LP}_\infty + D)$ Kripke complete?

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[9] By the way, analogously, $\mathbf{L}^c\mathbf{Fin}$ is the Kripke completion of the Kripke incomplete logic $(\mathbf{QH} + A^*_{\mathbf{Fin}} \& D)$, where $A^*_{\mathbf{Fin}}$ is the formula mentioned in Remark 2 (and introduced in [10]). The logic $(\mathbf{QH} + A^*_{\mathbf{Fin}})$ is again Kripke incomplete; its Kripke completion is $\mathbf{L}[\mathbf{Fin}^*]$, where $\mathbf{Fin}^*$ is the class of frames described by condition (φ) from our Remark 2, and this completion is not RE as well (cf. [10, Corollary 4]).

Question 2 Is the logic $\mathbf{LP}_\infty$ strongly Kripke complete?

Note that we are still unable to *directly* transfer our proof for $\mathbf{L}^c\mathbf{P}_\infty$ to $\mathbf{LP}_\infty$. Indeed, there exist rooted Kripke frames (M,U) with expanding domains such that: (i) $(\varnothing,\Delta^*)$ is satisfiable in (M,U) (in the root of M), i.e., $\mathbf{L}(M,U)\cap\Delta^*=\varnothing$,[10] and (ii) (M,U) validates $A^*_{\mathbf{P}_\infty}$ (i.e., it satisfies the condition described in Lemma 1). Namely, one can take a disjoint union of frames of all finite heights (e.g. the disjoint union of all finite chains S_n) with infinite domains (or with n-element domains X_n for all $n>0$) and add the root u_0 whose domain is finite (e.g. one-element). This means that unlike the proof of Claim (see in Section 2), now definitely one cannot guarantee that every frame (M,U) satisfying $(\varnothing,\Delta^*)$ falsifies $\mathbf{L}^c\mathbf{P}_\infty$ 'due to the formula $A^*_{\mathbf{P}_\infty}$'. On the other hand, we do not know if such Kripke frames (satisfying $(\varnothing,\Delta^*)$ and validating $A^*_{\mathbf{P}_\infty}$) validate the whole logic $\mathbf{LP}_\infty$ as well. In other words, we do not know, if $A^*_{\mathbf{P}_\infty}\in\mathbf{L}(M,U)$ implies $\mathbf{LP}_\infty\subseteq\mathbf{L}(M,U)$ (cf. the equivalence $(1)\Leftrightarrow(3)$ in Proposition 1).

This example is slightly related to the following open question:

Question 3 How to transfer (in a reasonable way) Lemma 2 and Corollary 2 to the case with expanding domains.[11]

3.3 To conclude this section, note that the formula $A^*_{\mathbf{Fin}}$ (see Remark 2) allows us to obtain the following variant of Theorem 2:

Theorem 3 *Let* $\mathbf{L}$ *be a predicate logic such that*

$$\mathbf{L}^c\mathbf{Fin} \subseteq \mathbf{L} \subseteq \bigcup(\,\mathbf{L}[\mathbf{Z}] : \mathbf{Z} \textit{ is an unrestricted class of Kripke frames}\,)$$

(here $\bigcup$ denotes the set-theoretical union, but not the sum of logics!).

Then $\mathbf{L}$ *is not strongly Kripke complete without parameters.*

Here a class $\mathbf{Z}$ of Kripke frames is called *unrestricted* (cf. [10, Section 1.2] or [11, Section 2.1]), if

$$\forall n\in\omega\ \exists(M,U)\in Z\ \exists u\in M\,(\mathrm{card}(M^u)\geq n,\ \mathrm{card}(U(u))\geq n).$$

The proof is similar to the proof of Theorem 2 given in Section 2; we use

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[10] More precisely, in order to satisfy the pair $(\varnothing,\Delta)$ in a frame (M,U), it is required to falsify all formulas from Δ by a single valuation in (M,U). However, this is not a serious problem. Our formulas C_m (for $m>1$) look rather regularly, and so, if cardinalities of domains are sufficiently large, it is easy to construct a valuation, which falsifies all these formulas (and similarly with P_n: $n>0$).

[11] While preparing the final text of this paper, we found an answer to this question. The proof of the corresponding claims for two different versions of Kripke semantics with finite expanding domains will be presented in [14]; on the other hand, for the third natural version of the semantics these properties do not hold.

the formula $A^*_{\mathbf{Fin}}$ for $A^*_{\mathbf{P}_\infty}$ and use the propositional formulas

$$\Phi_n = \bigvee_{i<j} (p_i \equiv p_j)$$

(in symbols $p_0, p_1, \ldots, p_n$) instead of P_n. □

Again all logics mentioned in Theorem 3 are not recursively axiomatizable, see Theorem from [10] and Theorem 2.1(2) from [11]; many of these logics are Π_2-arithmetical (cf. Section 2).

Also, for Kripke semantics with constant domains, we obtain the subsequent criterion of strong Kripke completeness for logics characterized by classes of finite posets (i.e., for logics of the form $\mathbf{L}^c\mathbf{Y}$ with $\mathbf{Y} \subseteq \mathbf{Fin}$); cf. our Corollary 1 in Section 2 and [10, Corollary 3]:

Proposition 2 *Let* $\mathbf{Y}$ *be a class of finite posets.*

(I) The following conditions are equivalent:

(1) *the logic* $\mathbf{L}^c\mathbf{Y}$ *is strongly Kripke complete (with or without parameters);*

(2) *the logic* $\mathbf{L}^c\mathbf{Y}$ *is recursively axiomatizable;*

(3) *the logic* $\mathbf{L}^c\mathbf{Y}$ *is finitely axiomatizable;*

(4) *the logic* $\mathbf{L}^c\mathbf{Y}$ *is 'tabular', i.e., it equals* $\mathbf{L}^c M$ *for a finite poset* M*;*

(5) $\mathbf{Y} \subseteq \mathbf{P}_n \cap \mathbf{W}_n$ *for some* $n > 0$*.*

(II) If $\mathbf{Y}$ *is a class of pairwise non-isomorphic finite rooted posets, then the mentioned conditions (1) – (5) are equivalent to*

(6) $\mathbf{Y}$ *is finite.*

Proof. The implication (1) ⇒ (5) readily follows from Theorem 3: namely, if $\forall n\, [\mathbf{Y} \not\subseteq \mathbf{P}_n \cap \mathbf{W}_n]$, then the class $\mathbf{Z}$ of all frames with constant domain over posets from $\mathbf{Y}$ is unrestricted. Similarly, the implication (2) ⇒ (5) is a consequence of Theorem from [10].

Clearly, (5) implies (4), because the family of (non-isomorphic) rooted posets from $\mathbf{P}_n \cap \mathbf{W}_n$ is finite, and $\mathbf{L}^c\mathbf{Y} = \mathbf{L}^c\mathbf{Y}' = \mathbf{L}^c M$, where $\mathbf{Y}'$ is the (finite) family of cones in posets from $\mathbf{Y}$ and M is the disjoint union of posets from $\mathbf{Y}'$.

Now, the implications (4) ⇒ (1) and (4) ⇒ (3) follow from Shimura's result [7, Theorem 3.7]. Indeed, let A be a formula axiomatizing the (finitely axiomatizable) propositional superintuitionistic logic of a finite poset M, and let $\mathbf{L} = [\mathbf{QH}{+}D\&A]$. The mentioned Shimura's theorem claims the strong Kripke completeness[12] of $\mathbf{L}$. From its proof we also obtain that $\mathbf{L} = \mathbf{L}^c M$.

Finally, the implication (3) ⇒ (2) is obvious. □

[12] naturally, in the usual sense, i.e., with parameters (cf. the beginning of Section 2)

Section 4. The proof of Main Lemma

Take a binary, unary, and 0-ary (i.e., propositional) symbols R, Q, p respectively. Take the following formulas:

$$
\begin{aligned}
A_0 &= \forall x \neg R(x,x) \,\&\, \forall x \exists y R(x,y) \,\&\, \forall x \forall y\, (R(x,y) \vee \neg R(x,y)) \,\&\\
&\quad \&\, \forall x \forall y \forall z\, (R(x,y) \& R(y,z) \supset R(x,z)),\\
A_1 &= \forall x \forall y\, [(\neg R(x,y) \vee p) \equiv ((Q(x) \supset Q(y)) \vee p)],\\
A_2 &= \forall x \forall y\, [(\neg R(x,y) \vee p) \equiv ((Q(y) \supset Q(x)) \vee p)],\\
A_3 &= \forall x \forall y\, [R(x,y) \& (Q(x) \supset Q(y)) \supset Q(x)],\\
A_4 &= \forall x \forall y\, [R(x,y) \& (Q(y) \supset Q(x)) \supset Q(y)],\\
A' &= A_0 \,\&\, [(A_1 \& A_3) \vee (A_2 \& A_4)];\\
A^* &= A' \supset p\,.
\end{aligned}
$$

4.1 IF PART. Suppose that $A^* \notin \mathbf{L}(M,U)$, i.e., $u \vDash A'$ and $u \nvDash p$ for some $u \in M$ and a valuation in (M^u, U); then we show that both $U(u)$ and $h[M^u]$ are infinite. Let us define the relation $(a < b) \Leftrightarrow (u \vDash R(a,b))$ on $U(u)$. By A_0, the relation is transitive, irreflexive, and there exists a sequence of different elements $a_0 < a_1 < \ldots$ in $U(u)$; hence $U(u)$ is infinite. Also $(a \not< b) \Leftrightarrow u \vDash \neg R(a,b)$ for $a, b \in U(u)$.

Now, let us assume that $h[M^u] = n$ is finite. Put $\Theta_i = \{v \in M^u \mid v \vDash Q(a_i)\}$ for $i \in \omega$.

First, let $u \vDash A_1 \& A_3$. Then, by A_1, for $i, j \in \omega$ we have:

$$(\Theta_i \subseteq \Theta_j) \Leftrightarrow u \vDash (Q(a_i) \supset Q(a_j)) \Leftrightarrow u \vDash \neg R(a_i, a_j) \Leftrightarrow a_i \not< a_j \Leftrightarrow j \leq i$$

(recall that $u \nvDash p$). Also, by A_3, if $i < j < k$, then

$$\forall v \in \Theta_i \backslash \Theta_j\, \exists w \in \Theta_j \backslash \Theta_k\, [v \leq w].$$

Indeed, if $j < k$ and $v \in \Theta_i \backslash \Theta_j$, then $v \nvDash Q(a_j)$ and $v \vDash R(a_j, a_k)$, so $v \nvDash (Q(a_j) \supset Q(a_k))$, i.e., $v \leq w$ for some $w \in \Theta_j \backslash \Theta_k$.

Hence we obtain an $(n+1)$-element chain $v_0 < v_1 < \ldots < v_n$ in M^u such that $v_i \in \Theta_i \backslash \Theta_{i+1}$ for $i < n$. This is a contradiction.

Second, let $u \vDash A_2 \& A_4$. Then, similarly, by A_2,

$$(\Theta_i \subseteq \Theta_j) \Leftrightarrow u \vDash \neg R(a_j, a_i) \Leftrightarrow i \leq j,$$

and by A_4, for $i < j < k$ we have

$$\forall v \in \Theta_k \backslash \Theta_j\, \exists w \in \Theta_j \backslash \Theta_i\, [v \leq w].$$

Hence we obtain an $(n+1)$-element chain $v_n < \ldots < v_0$, where $v_i \in \Theta_{i+1} \backslash \Theta_i$.

4.2 ONLY IF PART. Suppose that there exists $u \in M$ such that both $U(u)$ and $h[M^u]$ are infinite; then we show that $u \nvDash A^*$ for a suitable valuation on the cone (M^u, U).

Take a denumerable subset $X_0 = \{a_i \mid i \in \omega\}$ of $U(u)$. Let $\nu(a_i) = i$ for elements of X_0 and $\nu(a) = 0$ for all other elements from all $U(v)$, $v \geq u$.

First, assume that the cone M^u contains an ω-chain $u = u_0 < u_1 < \dots$. Then we define the following valuation on (M^u, U):

$$\begin{aligned} &v \vDash R(a,b) \iff \nu(a) < \nu(b); \\ &v \vDash Q(a) \iff v \not\leq u_{\nu(a)}; \\ &v \vDash p \iff v \neq u. \end{aligned}$$

Clearly, $u \vDash A_0$ and $u \nvDash p$. Also $u \vDash A_1$, because

$$u \vDash \neg R(a,b) \Leftrightarrow (\nu(b) \leq \nu(a)) \iff \forall v \geq u\, [\, v \vDash Q(a) \Rightarrow v \vDash Q(b)] \Leftrightarrow u \vDash (Q(a) \supset Q(b)),$$

and $v \vDash p$ for all $v > u$.

Now, show that $u \vDash A_3$. Assume that $a, b \in U(v)$, $v \vDash R(a,b) \& (Q(a) \supset Q(b))$, $v \nvDash Q(a)$. Then $\nu(a) < \nu(b)$ and $v \leq u_{\nu(a)} < u_{\nu(b)}$. Hence $v \nvDash (Q(a) \supset Q(b))$, since $u_{\nu(b)} \vDash Q(a)$, $u_{\nu(b)} \nvDash Q(b)$. This is a contradiction.

Therefore, $A^* \notin \mathbf{L}(M, U)$.

Second, assume, otherwise, that M^u does not contain ω-chains, i.e., M^u is a dually well-founded poset. Take $\Theta_i = \{v \in M^u \mid h[M^v] \leq i\}$ for $i < \omega$. Clearly, $\varnothing = \Theta_0 \subset \Theta_1 \subset \dots \subset \Theta_i \subset \Theta_{i+1} \subset \dots$ (recall that $h[M^u]$ is infinite). Take the following valuation on (M^u, U):

$$\begin{aligned} &v \vDash R(a,b) \iff \nu(a) < \nu(b); \\ &v \vDash Q(a) \iff v \in \Theta_{\nu(a)}; \\ &v \vDash p \iff v \neq u. \end{aligned}$$

Again $u \vDash A_0$ and $u \nvDash p$. Also $u \vDash A_2$, because

$$u \vDash (Q(b) \supset Q(a)) \iff (\Theta_{\nu(b)} \subseteq \Theta_{\nu(a)}) \iff (\nu(b) \leq \nu(a)) \iff u \vDash \neg R(a,b).$$

Finally, $u \vDash A_4$. Indeed, assume that $a, b \in U(v)$, $v \vDash R(a,b) \& (Q(b) \supset Q(a))$, $v \nvDash Q(b)$ for some $v \geq u$. Then $\nu(a) < \nu(b)$ and $v \notin \Theta_{\nu(b)}$, i.e., $h[M^v] > \nu(b)$. Then there exists $w \in M^v$ such that $h[M^w] = \nu(b)$ and so $w \in \Theta_{\nu(b)} \setminus \Theta_{\nu(a)}$, i.e., $w \vDash Q(b)$ and $w \nvDash Q(a)$. This contradicts $v \vDash (Q(b) \supset Q(a))$.

Hence again $A^* \notin \mathbf{L}(M, U)$. □

4.3 In conclusion, we can easily obtain the following variation of Lemma 1:

Lemma 3 *There exists a predicate sentence $A^*_{\mathbf{WF}_d}$ such that for every Kripke frame (M, U):* $A^*_{\mathbf{WF}_d} \in \mathbf{L}(M, U)$ *iff*

$\forall u \in M$ [*the domain $U(u)$ is finite, or the cone M^u is dually well-founded* (i.e., it does not contain ω-chains)].

Namely, put $A^*_{\mathbf{WF}_d} = A'' \supset p$, where $A'' = A_0 \& A_1 \& A_3$ (i.e., we drop the disjunct $A_2 \& A_4$ in the premise A' of $A^* = A^*_{\mathbf{P}_\infty}$).

The proof of this lemma repeats the proof for Main Lemma, with obvious changes. Namely, in the IF PART, using $A_1 \& A_3$, we can obtain an ω-chain

$v_0 < \ldots < v_i < \ldots$ in M^u such that $v_i \in \Theta_i \backslash \Theta_{i+1}$ for all $i \in \omega$ (the argument with $A_2 \& A_4$ is omitted now). And in the ONLY IF PART, we suppose that, for some $u \in M$, the domain $U(u)$ is infinite and the cone M^u contains an ω-chain $u = u_0 < u_1 < \ldots$; the second case, with dually well-founded M^u, is omitted. $\Box$

In particular, for Kripke frames with constant domains we have:

$$A^*_{\mathbf{WF}_d} \in \mathbf{L}(M, \lambda u.X) \;\; \textit{iff} \;\; [\; M \textit{ is dually well-founded, or } X \textit{ is finite} \;].$$

Note that here the restriction that M must be rooted (cf. footnote 5 in Section 2) is not required, because the class $\mathbf{WF}_d$ of dually well-founded posets, unlike $\mathbf{P}_\infty$ (and **Fin**), has the following convenient property:

$$\textit{Every poset } W \notin \mathbf{WF}_d \textit{ contains a cone } W^u \notin \mathbf{WF}_d. \qquad (c)$$

Clearly, $A^*_{\mathbf{WF}_d} \in \mathbf{L}\,\mathbf{WF}_d$, and so $A^*_{\mathbf{WF}_d} \& D \in \mathbf{L}^c\mathbf{WF}_d$.

Finally, Lemmas 1 and 3, together with [10, Lemma 1] (cf. our Remark 2 in Section 3), give the following chains of proper inclusions (here **PO** is the class of all posets):

Corollary 3 $\mathbf{QH} = \mathbf{L}\,\mathbf{PO} \subset \mathbf{L}\,\mathbf{WF}_d \subset \mathbf{L}\,\mathbf{P}_\infty \subset \mathbf{L}\,\mathbf{Fin}$ *and*
$[\mathbf{QH}{+}D] = \mathbf{L}^c\mathbf{PO} \subset \mathbf{L}^c\mathbf{WF}_d \subset \mathbf{L}^c\mathbf{P}_\infty \subset \mathbf{L}^c\mathbf{Fin}$.

In fact, $\mathbf{LWF}_d \not\subseteq [\mathbf{QH}{+}D]$, $\mathbf{LP}_\infty \not\subseteq \mathbf{L}^c\mathbf{WF}_d$, etc. Moreover, $\mathbf{LWF}_d \not\subseteq \mathbf{L}^c\mathbf{Y}$ for any class of posets $\mathbf{Y} \not\subseteq \mathbf{WF}_d$; thus $\mathbf{LY} \subset \mathbf{LWF}_d$ and $\mathbf{L}^c\mathbf{Y} \subset \mathbf{L}^c\mathbf{WF}_d$ for any class $\mathbf{Y} \supset \mathbf{WF}_d$ (the corresponding statements for **LFin** and for $\mathbf{LP}_\infty$ are given in Remark 2; they involve additional restrictions related to cones or to the rootedness, since the property (c) fails for **Fin** and for $\mathbf{P}_\infty$).

Remark 3 By the way, note that

$$\mathbf{L}\,\mathbf{WF} = \mathbf{L}\,\mathbf{PO} = \mathbf{QH} \quad \textit{and} \quad \mathbf{L}^c\mathbf{WF} = \mathbf{L}^c\mathbf{PO} = [\mathbf{QH}{+}D],$$

where **WF** is the class of well-founded posets, because the standard tree ω^* (the ω-branching tree of height ω) is well-founded, and **QH** (resp., $[\mathbf{QH}{+}D]$) is complete w.r.t. frames over ω^* (with expanding domains and with constant domains respectively), cf. [2, Theorem 6.4.17(1) and Proposition 7.6.15(1)].

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References

[1] Gabbay, D. and D. de Jongh, *Sequence of decidable finitely axiomatizable intermediate logics with the disjunction property*, Journal of Symbolic Logic **39**, No.1 (1974), pp. 67–78.

[2] Gabbay, D., V. Shehtman and D. Skvortsov, "Quantification in Nonclassical Logic Vol.1," Studies in Logic and the Foundation of Mathematics **153**, Elsevier, 2009.

[3] Ono, H., *A study of intermediate predicate logics*, Publications of RIMS, Kyoto Univ. **8**, No.3 (1972-1973), pp. 619–649.

[4] Ono, H., *Model extension theorem and Craig's interpolation theorem for intermediate predicate logics*, Reports on Math. Logic **15** (1983), pp. 41–58.

[5] Ono, H., *Some problems in intermediate predicate logics*, Reports on Math. Logic **21** (1987), pp. 55–67.

[6] Shehtman, V., *On strong neighbourhood completeness of modal and intermediate propositional logics, 1*, in: *Advances in Modal Logic '96*, M. Kracht, M. De Rijke, H. Wansing, M. Zakharyaschev, eds., CSLI Publications, 1997, pp. 209–222.

[7] Shimura, T., *Kripke completeness of some intermediate predicate logics with the axiom of constant domain and a variant of canonical formulas*, Studia Logica **52**, No.1 (1993), pp. 23–40.

[8] Shimura, T., *On completeness of intermediate predicate logics with respect to Kripke semantics*, Bulletin of the Section of Logic **24**, No.1 (1995), pp. 41–45.

[9] Skvortsov, D., *On axiomatizability of some intermediate predicate logics (summary)*, Reports on Math. Logic **22** (1988), pp. 115–116.

[10] Skvortsov, D., *The predicate logic of finite Kripke frames is not recursively axiomatizable*, Journal of Symbolic Logic **70** (2005), pp. 451–459.

[11] Skvortsov, D., *On non-axiomatizability of superintuitionistic predicate logics of some classes of well-founded and dually well-founded Kripke frames*, Journal of Logic and Computation **16**, No.5 (2006), pp. 685–695.

[12] Skvortsov, D., *On non-axiomatizability of some superintuitionistic predicate logics, I: Predicate logics of well-ordered and dually well-ordered Kripke frames*, in preparation.

[13] Skvortsov, D., *A new (simpler) solution to Ono's problem on the strong completeness for intermediate predicate logics*, in: *Third St.Petersburg Days of Logic and Computability*, St.Petersburg, Russia, Abstracts, 2015, p. 20.

[14] Skvortsov, D., *On superintuitionistic predicate logics of Kripke frames with finite domains, I: Basic notions and the finite model property*, in preparation.

[15] Smoryński, C., *Applications of Kripke models*, in: *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis* (ed. A.S. Troelstra), Lecture Notes Math. **344**, Springer, 1973, pp. 324–391.

[16] Yokota, S., *Axiomatization of the first-order intermediate logics of bounded Kripkean heights, I*, Zeitschr. für math. Logik und Grundl. der Math. **35**, No.5 (1989), pp. 415–421.