Relational Semantics for the Turing Schmerl Calculus

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Abstract

In [13] the authors introduced the propositional modal logic **TSC** (which stands for Turing Schmerl Calculus) which adequately describes the provable interrelations between different kinds of Turing progressions. The current paper defines a model $\mathcal{J}$ which is proven to be a universal model for **TSC**. The model $\mathcal{J}$ is a slight modification of the intensively studied $\mathcal{I}$: Ignatiev's universal model for the closed fragment of Gödel Löb's polymodal provability logic **GLP**.

Keywords: Provability logic, strictly positive fragments, Turing progressions, universal model.

1 Introduction

Turing progressions arise by iteratedly adding consistency statements to a base theory. Different notions of consistency give rise to different Turing progressions. In [13], the authors introduced the system **TSC** (sometimes denoted by Cyrillic letter **Tse** and Latin **C**) that generates exactly all relations that hold between these different Turing progressions given a particular set of natural consistency notions. The system was proven to be arithmetically sound and complete for a natural interpretation, named the *Formalized Turing progressions* (FTP) interpretation. A brief overview of this work can be found in Section 2.1 together with Theorem 3.8.

In this paper we discuss relational semantics of **TSC** by considering a small modification on Ignatiev's frame, which is a universal frame for the variable-free fragment of Japaridze's provability logic **GLP**.

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2 Strictly positive signature

TSC is built-up from a positive propositional modal signature using *ordinal modalities*. Let Λ be a fixed recursive ordinal throughout the paper with some properties as specified in Remark 3.4. By ordinal modalities we denote modalities of the form $\langle n^\alpha \rangle$ where $\alpha \in \Lambda$ and $n \in \omega$ (named *exponent* and *base*, respectively). The set of formulas in this language is defined as follows:

Definition 2.1 By $\mathbb{F}$ we denote the smallest set such that:

i) $\top \in \mathbb{F}$;
ii) If $\varphi,\, \psi \in \mathbb{F} \Rightarrow (\varphi \wedge \psi) \in \mathbb{F}$;
iii) if $\varphi \in \mathbb{F},\ n < \omega$ and $\alpha < \Lambda \Rightarrow \langle n^\alpha \rangle\varphi \in \mathbb{F}$.

For any formula ψ in this signature, we define the set of base elements occurring in ψ. That is:

Definition 2.2 The set of base elements occurring in any modality of a formula $\psi \in \mathbb{F}$ is denoted by $\mathsf{N\text{-}mod}(\psi)$. We recursively define $\mathsf{N\text{-}mod}$ as follows:

i) $\mathsf{N\text{-}mod}(\top) = \emptyset$;
ii) $\mathsf{N\text{-}mod}(\varphi \wedge \psi) = \mathsf{N\text{-}mod}(\varphi) \cup \mathsf{N\text{-}mod}(\psi)$;
iii) $\mathsf{N\text{-}mod}(\langle n^\alpha \rangle\psi) = \{n\} \cup \mathsf{N\text{-}mod}(\psi)$.

2.1 (FTP) interpretation

In [13], the authors introduced an arithmetical interpretation in which modal formulas are intended to be read as *Turing progressions*; hierarchies of theories that arise by transfinitely iterating n-consistency statements. These progressions can be defined according to the following conditions below:

T1. $(T)^0_n := T$ where T is an initial or base theory;

T2. $(T)^{\alpha+1}_n := (T)^\alpha_n \cup \{\mathrm{Con}_n((T)^\alpha_n)\}$;

T3. $(T)^\lambda_n := \bigcup_{\beta<\lambda}(T)^\beta_n$, for λ a limit ordinal below Λ.

However, conditions T1-T3 can be reformulated by the unique following clause:

$$(T)^\alpha_n := T \cup \{\mathrm{Con}_n((T)^\beta_n) : \beta < \alpha\} \quad \text{for } \alpha < \Lambda,\ n < \omega.$$

This presentation, known as *Smooth Turing progressions*, was studied by Beklemishev in among others [5] and [1].

Given such a family of theories $(T)^\alpha_n$ we have that they can be represented within $\mathbf{EA}^+$ through some arithmetical formula numerating their axioms. Here, $\mathbf{EA}^+$ is Robinson's arithmetic $\mathbf{Q}$ together with induction for bounded formulas.

Suppose we are given some elementary well-ordering $(D, \prec)$. Consider the elementary formula $\tau^{\sigma(z)}_n(x, y)$ where x is a variable for an ordinal $\alpha \in D$, y stands for the coding of some arithmetical formula and $\sigma(z)$ is an elementary formula enumerating the axioms of some base theory. Hence, roughly speaking, the formula tells us that the formula coded by y is an axiom of $(T)^\alpha_n$ where the initial theory is numerated by the elementary formula $\sigma(z)$ and $\mathbf{EA}^+$ is numerated by $\epsilon(x)$.

We say that $\tau_n^{\sigma(z)}(\alpha, x)$ enumerates the α-th theory of a progression based on iteration of consistency along $(D, \prec)$ with base $\sigma(z)$ if:

$$\mathbf{EA}^+ \vdash \tau_n^{\sigma(z)}(\alpha, x) \leftrightarrow ((\epsilon(x) \vee \sigma(x)) \vee \exists \beta \,(\prec(\beta, \alpha) \ \wedge x = \ulcorner \mathrm{Con}_n(\tau_n^{\sigma(z)}(\dot{\beta}, y) \urcorner)).$$

The existence of such $\tau_n^{\sigma(z)}(\alpha, x)$ is guaranteed by the fixed point theorem.

Let us introduce now the arithmetical interpretation of our modal formulae in terms of the τ-formulae. Let $\mathcal{L}_{\mathbb{N}}$ denote the set of formulas in the usual language of arithmetic.

Definition 2.3 An arithmetical interpretation is a map $* : \mathbb{F} \longrightarrow \mathcal{L}_{\mathbb{N}}$ inductively defined as follows:

(i) $(\top)^*(x) = \epsilon(x)$;
(ii) $(\varphi \wedge \psi)^*(x) = (\varphi)^*(x) \vee (\psi)^*(x)$
(iii) $(\langle n^\alpha \rangle \varphi)^*(x) = \tau_n^{\varphi^*(y)}(\alpha, x)$.

Since $\mathbb{F}$ has no propositional variables, we can identify a modal formula with its arithmetical interpretation unambiguously. Moreover, for the sake of clarity, and since we are working in the close fragment, we will use the following notation: given $\varphi \in \mathbb{F}$ by Th_φ we denote Th_σ where $\varphi^*(x) = \sigma(x)$, following Definition 2.3. If $\varphi^*(x) = \epsilon(x)$ we use just $\mathbf{EA}^+$ instead of Th_ϵ.

3 The logic TSC

In this section we introduce the logic **TSC** whose main goal is to express valid relations that hold between the corresponding Turing progressions. For this purpose we shall consider a kind of special formulas named *monomial normal forms* which are used in the axiomatization of the calculus **TSC**.

Monomial normal forms are conjunctions of monomials with an additional condition on the occurring exponents. In order to formulate this condition we first need to define the *hyper-exponential* as studied in [9].

Definition 3.1 For every $n \in \omega$ the *hyper-exponential* functions $e^n : \mathrm{On} \to \mathrm{On}$ are recursively defined as follows: e^0 is the identity function, $e^1 : \alpha \mapsto -1 + \omega^\alpha$ and $e^{n+m} = e^n \circ e^m$.

We will use e to denote e^1. Note that for α not equal to zero we have that $e(\alpha)$ coincides with the regular ordinal exponentiation with base ω; that is, $\alpha \mapsto \omega^\alpha$. However, it turns out that hyper-exponentials have the nicer algebraic properties in the context of provability logics.

The next definition may seem a bit ad-hoc in a purely syntactical setting so we provide some minimal motivation. Monomials are terms of the form $\langle n^\alpha \rangle \top$, for $n < \omega$ and $\alpha < \Lambda$. The simplest form of stating information about Turing progressions will be by means of a conjunction of monomials. However, the arithmetical behavior of monomials tells us that monomials will imply other monomials. In our normal form, we wish to only include those monomials that add new information to the entire expression giving rise to the technical Condition c below.

Definition 3.2 The set of formulas in *monomial normal form*, MNF, is inductively defined as follows:

i) $\top \in \mathsf{MNF}$;

ii) $\langle n^{\alpha} \rangle \top \in \mathsf{MNF}$, for any $n < \omega$ and $\alpha < \Lambda$;

iii) if *a)* $\langle n_0^{\alpha_0} \rangle \top \wedge \ldots \wedge \langle n_k^{\alpha_k} \rangle \top \in \mathsf{MNF}$;

b) $n < n_0$;

c) α of the form $e^{n_0 - n}(\alpha_0) \cdot (2 + \delta)$ for some $\delta < \Lambda$,

then $\langle n^{\alpha} \rangle \top \wedge \langle n_0^{\alpha_0} \rangle \top \wedge \ldots \wedge \langle n_k^{\alpha_k} \rangle \top \in \mathsf{MNF}$.

The derivable objects of **TSC** are *sequents* i.e. expressions of the form $\varphi \vdash \psi$ where $\varphi, \psi \in \mathbb{F}$. We will use the following notation: by $\varphi \equiv \psi$ we will denote that both $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are derivable. Also, by convention we take that for any n, $\langle n^0 \rangle \varphi$ is just φ.

Definition 3.3 **TSC** is given by the following set of axioms and rules:

Axioms:

(i) $\varphi \vdash \varphi, \quad \varphi \vdash \top$;

(ii) $\varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi$;

(iii) Monotonicity axioms: $\langle n^{\alpha} \rangle \varphi \vdash \langle n^{\beta} \rangle \varphi$, for $\beta < \alpha$;

(iv) Co-additivity axioms: $\langle n^{\beta+\alpha} \rangle \varphi \equiv \langle n^{\alpha} \rangle \langle n^{\beta} \rangle \varphi$;

(v) Reduction axioms: $\langle (n+m)^{\alpha} \rangle \varphi \vdash \langle n^{e^m(\alpha)} \rangle \varphi$;

(vi) Schmerl axioms:

$$\langle n^{\alpha} \rangle \big(\langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi \big) \equiv \langle n^{e^{n_0 - n}(\alpha_0) \cdot (1+\alpha)} \rangle \top \wedge \langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi$$

for $n < n_0$ and $\langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi \in \mathsf{MNF}$.

Rules:

(i) If $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;

(ii) If $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;

(iii) If $\varphi \vdash \psi$, then $\langle n^{\alpha} \rangle \varphi \vdash \langle n^{\alpha} \rangle \psi$;

(iv) If $\varphi \vdash \psi$, then $\langle n^{\alpha} \rangle \varphi \wedge \langle m^{\beta+1} \rangle \psi \vdash \langle n^{\alpha} \rangle \big(\varphi \wedge \langle m^{\beta+1} \rangle \psi \big)$ for $n > m$.

It is worth mentioning the special character of Axioms (v) and (vi) since both axioms are modal formulations of principles related to Schmerl's fine structure theorem, also known as *Schmerl's formulas* (see [16] and [3]).

Remark 3.4 As we see in the axioms of our logic, they only make sense if the ordinals occuring in them are available. Recall that Λ is fixed to be a recursive ordinal all through the paper. Moreover, some usable closure conditions on Λ naturally suggest themselves. Since it suffices to require that for $n < \omega$ that $\alpha, \beta < \Lambda \;\Rightarrow\; \alpha + e^n(\beta) < \Lambda$, we shall for the remainder assume that Λ is an ε-number, that is, a positive fixpoint of e whence $e(\Lambda) = \Lambda = \omega^\Lambda$.

In [13], the authors proved that for any formula φ, there is a unique equivalent ψ in monomial normal form.

Theorem 3.5 *For every formula φ there is a unique $\psi \in \mathsf{MNF}$ such that $\varphi \equiv \psi$.*

In virtue of the Reduction axioms, a formula $\psi \in \mathsf{MNF}$ may bear implicit information on monomials $\langle n^\alpha \rangle\top$ for $n \notin \mathsf{N\text{-}mod}(\psi)$. The next definition is made to retrieve this information.

Definition 3.6 Let $\psi := \langle n_0^{\alpha_0} \rangle\top \wedge \ldots \wedge \langle n_k^{\alpha_k} \rangle\top \in \mathsf{MNF}$. By $\pi_{n_i}(\psi)$ we denote the corresponding exponent α_i. Moreover, for $m \notin \mathsf{N\text{-}mod}(\psi)$, with $n_k > m$, $\pi_m(\psi)$ is set to be $e\big(\pi_{m+1}(\psi)\big)$ and for $m' > n_k$, $\pi_{m'}(\psi)$ is defined to be 0.

The following theorems are proven in [13]. The first one provides a succinct derivability condition between monomial normal forms while the second one establishes the soundness and completeness of the system with respect to the (FTP) interpretation:

Theorem 3.7 *For any ψ_0, $\psi_1 \in \mathsf{MNF}$, where $\psi_0 := \langle n_0^{\alpha_0} \rangle\top \wedge \ldots \wedge \langle n_k^{\alpha_k} \rangle\top$ and $\psi_1 := \langle m_0^{\beta_0} \rangle\top \wedge \ldots \wedge \langle m_j^{\beta_j} \rangle\top$. We have that $\psi_0 \vdash \psi_1$ iff for any $n < \omega$, $\pi_n(\psi_0) \geq \pi_n(\psi_1)$.*

Theorem 3.8 *For any φ, $\psi \in \mathbb{F}$,*

$$\varphi \vdash \psi \Longleftrightarrow \boldsymbol{EA}^+ \vdash \forall x \; \big(\Box_{\mathsf{Th}_\psi}(x) \to \Box_{\mathsf{Th}_\varphi}(x)\big).$$

4 A variation on Ignatiev's Frame

The purpose of this section is to define a modal model $\mathcal{J}$ which is universal for our logic. That is, any derivable sequent will hold everywhere in the model whereas any non-derivable sequent will be refuted somewhere in the model.

The model will be based on special sequences of ordinals. In order to define them, we need the following central definition.

Definition 4.1 We define *ordinal logarithm* as $\ell(0) := 0$ and $\ell(\alpha + \omega^\beta) := \beta$.

With this last definition we are now ready to introduce the set of worlds of our frame.

Definition 4.2 By Ig^ω we denote the set of *ℓ-sequences* or *Ignatiev sequences*. That is, the set of sequences $x := \langle x_0, x_1, x_2, \ldots \rangle$ where for $i < \omega$, $x_{i+1} \leq \ell(x_i)$.

Given a ℓ-sequence x, if all but finitely many of its elements are zero, we will write $\langle x_0, \ldots, x_n, \mathbf{0} \rangle$ to denote such ℓ-sequence or even simply $\langle x_0, \ldots, x_n \rangle$ whenever $x_{n+1} = 0$.

Next, we can define our frame, which is a variation of Ignatiev's frame.

Definition 4.3 $\mathcal{J}_\Lambda := \langle I, \{R_n\}_{n<\omega} \rangle$ is defined as follows:

$$I := \{x \in \mathrm{Ig}^\omega : x_i < \Lambda \text{ for } i < \omega\}$$

and

$$xR_n y :\Leftrightarrow (\forall m \leq n \; x_m > y_m \; \wedge \; \forall i > n \; x_i \geq y_i).$$

Since Λ is a fixed ordinal along the paper, from now on we suppress the subindex Λ.

The observations collected in the next lemma all have elementary proofs. Basically, the lemma confirms that the R_n are good to model provability logic and respect the increasing strength of the provability predicates $[n]$.

Lemma 4.4

(i) *Each R_n for $n \in \omega$ is* transitive*: $xR_n y \; \wedge \; yR_n z \; \Rightarrow \; xR_n z$;*

(ii) *Each R_n for $n \in \omega$ is* Noetherian*: each non-empty $X \subseteq I$ has an R_n-maximal element $y \in X$, i.e., $\forall x{\in}X \; \neg yR_n x$;*

(iii) *The relations R_n are* monotone *in n in the sense that: $xR_n y \Rightarrow xR_m y$ whenever $n > m$.*

Note that Item (ii) is equivalent to stating that there are no infinite ascending R_n chains. In other words, the converse of R_n is well-founded.

We define the auxiliary relations R_n^α for any $n < \omega$ and $\alpha < \Lambda$. The idea is that the R_n^α will model the $\langle n^\alpha \rangle$ modality.

Definition 4.5 Given $x, y \in I$ and R_n on I, we recursively define $xR_n^\alpha y$ as follows:

(i) $xR_n^0 y \quad :\Leftrightarrow \quad x = y$;

(ii) $xR_n^{1+\alpha} y \; :\Leftrightarrow \; \forall \beta{<}1{+}\alpha \; \exists z \; \big(xR_n z \; \wedge \; zR_n^\beta y\big)$.

Let us introduce some simple observations about the R_n^α relations.

Proposition 4.6 *Given $x, y \in I$, $n < \omega$ and $\alpha < \Lambda$:*

$$xR_n^{\alpha+1} y \; \Leftrightarrow \; \exists z \; \big(xR_n z \; \wedge \; zR_n^\alpha y\big).$$

Proof We make a case distinction on α with $\alpha = 0$ being trivial. For $\alpha > 0$, we have that $\alpha = 1 + \gamma$ for some $\gamma \leq \alpha$. With the help of this fact, we can reason as follows:

$$\begin{aligned} xR_n^{\alpha+1} y \; &\Leftrightarrow \; xR_n^{1+\gamma+1} y; \\ &\Leftrightarrow \; \forall \beta < 1 + \delta + 1 \, \exists z \; \big(xR_n z \; \wedge \; zR_n^\beta y\big); \\ &\Rightarrow \; \exists z \big(xR_n z \; \wedge \; zR_n^\alpha y\big), \text{ in particular.} \end{aligned}$$

Thus, $xR_n^{\alpha+1}y \Rightarrow \exists z\big(xR_nz \wedge zR_n^{\alpha}y\big)$. For right-to-left implication we proceed analogously:

$$\begin{aligned}
\exists z\big(xR_nz \ \wedge\ zR_n^{\alpha}y\big) \ &\Leftrightarrow\ \exists z\ \big(xR_nz \ \wedge\ zR_n^{1+\gamma}y\big);\\
&\Leftrightarrow\ \exists z\ \big(xR_nz \ \wedge\ zR_n^{1+\gamma}y\big) \wedge\\
&\qquad \forall\beta < 1+\gamma\ \exists z'\ \big(zR_nz' \ \wedge\ z'R_n^{\beta}y\big);\\
&\Rightarrow \forall\beta' < 1+\gamma+1\ \exists u\ \big(xR_nu \ \wedge\ uR_n^{\beta'}y\big);\\
&\Leftrightarrow\ xR_n^{1+\gamma+1}y;\\
&\Leftrightarrow\ xR_n^{\alpha+1}y.
\end{aligned}$$

□

Proposition 4.7 *Let $x, y \in I$, $n < \omega$ and $\lambda < \Lambda$ such that $\lambda \in Lim$:*

$$xR_n^{\lambda}y \ \Leftrightarrow\ \forall\beta < \lambda\ xR_n^{1+\beta}y.$$

Proof For left-to-right implication, notice that if $xR_n^{\lambda}y$ then by definition, we have that $\forall\beta < \lambda\ \exists u\ \big(xR_nu \ \wedge\ uR_n^{\beta}y\big)$. Therefore, in particular, we obtain that $\forall\beta < \lambda\ \exists u\ \big(xR_nu \ \wedge\ uR_n^{1+\beta}y\big)$ thus by transitivity, $\forall\beta < \lambda\ \ xR_n^{1+\beta}y$. For the other direction, if $\forall\beta < \lambda\ xR_n^{1+\beta}y$, then in particular, $\forall\beta < \lambda\ xR_n^{\beta+1}y$ and then, by Proposition 4.6, $\forall\beta < \lambda\ \exists u\ \big(xR_nu \ \wedge\ uR_n^{\beta}y\big)$, that is, $xR_n^{\lambda}y$. □

It is easy to see that for example $\langle\omega, \mathbf{0}\rangle R_0^n\langle m, \mathbf{0}\rangle$ for each $n, m \in \omega$, so that also $\langle\omega, \mathbf{0}\rangle R_0^{\omega}\langle m, \mathbf{0}\rangle$ for each $m \in \omega$. Clearly, we do not have $\langle\omega, \mathbf{0}\rangle R_0^{\omega+1}\langle m, \mathbf{0}\rangle$ for any $m \in \omega$ but we do have $\langle\omega+1, \mathbf{0}\rangle R_0^{\omega+1}\langle m, \mathbf{0}\rangle$ for all $m \in \omega$.

We also note that the dual definition $x\overline{R}_n^0y \ :\Leftrightarrow\ x = y$; and $x\overline{R}_n^{1+\alpha}y \ :\Leftrightarrow \forall\,\beta{<}1{+}\alpha\ \exists z\ \big(x\overline{R}_n^{\beta}z \ \wedge\ z\overline{R}_ny\big)$ does not make much sense on our frames. For example we could have $\langle\omega, \mathbf{0}\rangle\overline{R}_0^{\alpha}\langle 0, \mathbf{0}\rangle$ for any ordinal $\alpha > 0$.

With the the auxiliary relations R_n^{α}, we give the following definition for a formula φ being true in a point x of $\mathcal{J}$.

Definition 4.8 Let $x \in I$ and $\varphi \in \mathbb{F}$. By $x \Vdash \varphi$ we denote the validity of φ in x that is recursively defined as follows:

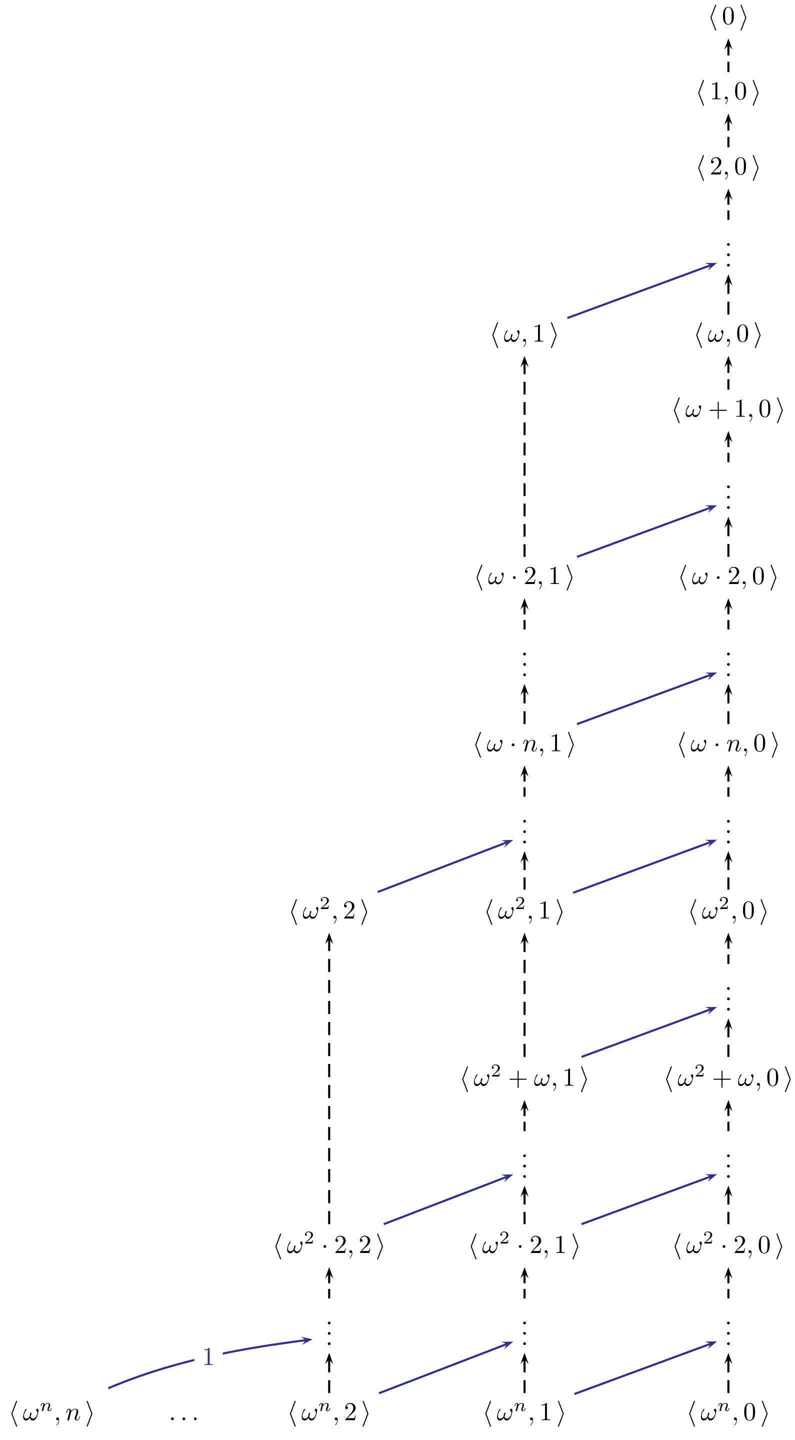
- $x \Vdash \top$ for all $x \in I$;
- $x \Vdash \varphi \wedge \psi$ iff $x \Vdash \varphi$ and $x \Vdash \psi$;
- $x \Vdash \langle n^{\alpha}\rangle\varphi$ iff there is $y \in I$, $xR_n^{\alpha}y$ and $y \Vdash \varphi$.

Here are some easy observations on the R_n^{α} relations which among others tell us that all the R_n^{α} serve the purpose of a provability predicate for any $n \in \omega$ and $\alpha < \Lambda$.

Lemma 4.9

(i) *Each $R_n^{1+\alpha}$ for $n \in \omega$ and α an ordinal is* transitive*: $xR_n^{1+\alpha}y \wedge yR_n^{1+\alpha}z \Rightarrow xR_n^{1+\alpha}z$;*

Figure 1. A fragment of our frame $\mathcal{J}$. The dashed arrows represent R_0 relations, while the continuous arrows represent R_1 relations.



(ii) *Each* $R_n^{1+\alpha}$ *for* $n \in \omega$ *and* α *an ordinal is* Noetherian*: each non-empty* $X \subseteq I$ *has an* $R_n^{1+\alpha}$*-maximal element* $y \in X$*, i.e.,* $\forall x{\in}X\ \neg y R_n^{1+\alpha} x$*;*

(iii) *The relations* $R_n^{1+\alpha}$ *are* monotone *in* n *in the sense that:* $xR_n^{1+\alpha}y \Rightarrow xR_m^{1+\alpha}y$ *whenever* $n > m$*;*

(iv) *The relations* $R_n^{1+\alpha}$ *are* monotone *in* $1+\alpha$ *in the sense that:* $xR_n^{1+\alpha}y \Rightarrow xR_n^{1+\beta}y$ *whenever* $1+\beta < 1+\alpha$*.*

Proof The first three items follow directly from Lemma 4.4 by an easy transfinite induction. The last item is also easy. □

5 A characterization for transfinite accessibility

The intuitive idea behind the $xR_n^{\alpha}y$ assertion, is that this tells us that there exists a chain of 'length' α of R_n steps leading from the point x up to the point y. The following useful lemma tries to capture this intuition.

Lemma 5.1 *For* $x, y \in I$ *and* $n < \omega$ *we have that the following are equivalent*

(i) $xR_n^{1+\alpha}y$

(ii) *For each* $\beta < 1+\alpha$ *there exists a collection* $\{x^\gamma\}_{\gamma<\beta}$ *so that*
   (a) xR_nx^γ *for any* $\gamma < \beta$*,*
   (b) $x^0 = y$ *and,*
   (c) *for any* $\gamma' < \gamma < \beta$ *we have* $x^\gamma R_n x^{\gamma'}$*.*

Proof By induction on α. □

We shall now provide a characterization of the $R_n^{1+\alpha}$ relations. To this end, let us for convenience define

$$xR_{-1}^{\zeta}y \quad :\Leftrightarrow \quad \forall\, n{>}0 \ \ x_n \geq y_n.$$

With this notation the following theorem makes sense.

Theorem 5.2 *For* $x, y \in I$ *and* $n < \omega$ *we have that the following are equivalent*

(i) $xR_n^{1+\alpha}y$*;*

(ii) $x_n \geq y_n + \big(1 + e(y_{n+1})\big) \cdot (1+\alpha)$ *and* $xR_{n-1}^{e(1+\alpha)}y$*;*

(iii)
$$\begin{array}{ll} x_n \geq y_n + \big(1 + e(y_{n+1})\big) \cdot (1+\alpha) & \textit{and,} \\ x_m > y_m \textit{ for } m < n & \textit{and,} \\ x_m \geq y_m \textit{ for } m > n. & \end{array}$$

We dedicate the remainder of this section to proving this theorem and move there through a series of lemmas. The first lemma in this series is pretty obvious. It tells us that if we can move from x to y in α many steps, then the distance between x_n and y_n must allow α many steps; That is, they lie at least α apart.

Lemma 5.3 *For* $x, y \in I$ *and* $n < \omega$ *and any ordinal* $\alpha < \Lambda$*, if* $xR_n^{\alpha}y$ *then* $x_n \geq y_n + \alpha$*.*

Proof By an easy induction on α. □

However, how many R_n steps one can make is not entirely determined by the n coordinates of the points. For example, there is just a single R_0 step from the point $\langle \omega \cdot 2, 1 \rangle$ to the point $\langle \omega, 1 \rangle$ whereas these points lie ω apart on the '0 coordinate'. The following lemma tells us how for R_n steps, the n-th coordinates are affected by the values of the $n+1$-th coordinate.

Lemma 5.4 *For $x, y \in I$ and $n < \omega$ with $xR_n^{1+\alpha}y$, we have*

$$x_n \geq y_n + e(y_{n+1}) \cdot (1+\alpha).$$

In order to give a smooth presentation of this proof, we first give two simple technical lemmas with useful observations on the ordinals and ordinal functions involved.

Lemma 5.5 *For α, β and γ ordinals we have*

(i) $\ell(\beta) \geq 1+\alpha \iff \beta \in e(1+\alpha) \cdot (1+\mathsf{On})$,

(ii) *If* $(1+\alpha) < \beta$ *and* $\gamma \in e(\beta) \cdot (1+\mathsf{On})$, *then* $\gamma \in e(1+\alpha) \cdot (1+\mathsf{On})$,

(iii) $e(\beta + (1+\alpha)) = e(\beta) \cdot e(1+\alpha)$,

(iv) *For α a limit ordinal, we have that*

$$xR_n^{\alpha}y \iff \forall\, 1{+}\beta{<}\alpha\, \exists z\ (xR_n z \wedge zR_n^{1+\beta}y).$$

Proof The first two items then can easily be seen by using a Cantor Normal Form expression with base ω. For Item (i), we use the fact that $\beta \in \mathrm{Lim}$ together with that if $\ell(\beta) \geq 1+\alpha$, then $\beta \geq e(\ell(\beta)) \geq e(1+\alpha)$. For Items (ii) and (iii) we use that $e(1+\omega) = \omega^{1+\omega} = \omega^1 \cdot \omega^\omega$. The last item follows from Definition 4.5 together with the fact that $1+\alpha = \alpha \in \mathrm{Lim}$. □

Lemma 5.6 *For $x, y \in I$ and $n < \omega$, $xR_n y \Longrightarrow x_n \geq y_n + e(x_{n+1})$.*

Proof We make a case distinction on x_n. If $x_n \in \mathrm{Succ}$ then is trivial since $e(x_{n+1}) = 0$. If $x_n \in \mathrm{Lim}$, and furthermore, x_n is an additively indecomposable limit ordinal, it follows from the fact that $x_n > y_n$ and $x_n \geq e(x_{n+1})$. Otherwise, we can rewrite x_n as $\alpha + e(\beta)$ for some $\beta \geq x_{n+1}$, and y_n as $\delta + \omega^\gamma$. If $y_n \leq \alpha$ then clearly $x_n \geq y_n + e(x_{n+1})$. If $\alpha = \delta$ and $\gamma < \beta$, then notice that $\omega^\gamma + e(\beta) = e(\beta)$ Thus, we have that $\alpha + e(\beta) = \delta + \omega^\gamma + e(\beta) \geq y_n + e(x_{n+1})$. □

With these technical lemmas at hand we can now prove Lemma 5.4.

Proof By induction on α. For $\alpha := 0$, we check that $x_n \geq y_n + e(y_{n+1})$. Note that since $xR_n y$ then $x_n \geq y_n + e(x_{n+1})$ and $x_{n+1} \geq y_{n+1}$, then $x_n \geq y_n + e(y_{n+1})$. For $\alpha := \beta + 1$, if $xR_n^{1+\beta+1}y$ then there is $z \in I$ such that $xR_n z$ and $zR_n^{1+\beta}y$. Thus, we have the following:

(i) $x_n \geq z_n + e(z_{n+1})$;

(ii) $z_n \geq y_n + e(y_{n+1}) \cdot (1+\beta)$.

Therefore, $x_n \geq y_n + e(y_{n+1}) \cdot (1+\beta) + e(z_{n+1})$. Since $e(z_{n+1}) \geq e(y_{n+1})$ then $x_n \geq y_n + e(y_{n+1}) \cdot (1+\beta) + e(y_{n+1})$ i.e. $x_n \geq y_n + e(y_{n+1}) \cdot (1+\beta+1)$. For $\alpha \in \mathrm{Lim}$, notice that by IH, we have that $x_n \geq y_n + e(y_{n+1}) \cdot (1+\delta)$ for $\delta < \alpha$. Thus, $x_n \geq y_n + e(y_{n+1}) \cdot (1+\alpha)$. $\square$

Combining Lemma 5.4 and Lemma 5.3 we get the following.

Corollary 5.7 *For $x, y \in I$ and $n < \omega$ we have that*

$$xR_n^{1+\alpha}y \Rightarrow x_n \geq y_n + \big(1 + e(y_{n+1})\big) \cdot (1+\alpha).$$

This corollary takes care of part of the implication from Item (i) to Item (ii) in Theorem 5.2. We will now focus on the implication from Item (iii) to Item (i) but before we do so, we first formulate a simple yet useful lemma.

Lemma 5.8 *For $x, y \in I$, if $xR_{m+1}y$, then $x_m \geq y_m + e(x_{m+1})$.*

Proof Since R_{m+1} is contained in R_m, if $xR_{m+1}y$ then xR_my and thus by Lemma 5.6, $x_m \geq y_m + e(x_{m+1})$. $\square$

With this technical lemma we can obtain the next step in the direction from Item (iii) to Item (i) in Theorem 5.2.

Lemma 5.9 *For $x, y \in I$ and $n < \omega$ we have that if*

$$\begin{array}{ll} x_n \geq y_n + \big(1 + e(y_{n+1})\big) \cdot (1+\alpha) & \textit{and,} \\ x_m > y_m \textit{ for } m < n & \textit{and,} \\ x_m \geq y_m \textit{ for } m > n. & \end{array}$$

then

$$xR_n^{1+\alpha}y.$$

Proof We use Lemma 5.1 whence are done if we can find for each $\beta < 1+\alpha$ there exists a collection $\{x^\gamma\}_{\gamma<\beta}$ so that

(i) xR_nx^γ for any $\gamma < \beta$,

(ii) $x_0 = y$ and,

(iii) for any $\gamma' < \gamma < \beta$ we have $x^\gamma R_n x^{\gamma'}$.

We define x^γ uniformly as follows. We define $x^0 := y$ and

$$x_m^{1+\gamma} \quad := \quad \begin{cases} y_m & \text{in case } m > n, \\ y_m + \big(1 + e(y_{n+1})\big) \cdot (1+\gamma) & \text{in case } m = n, \\ y_m + e(y_{m+1}) & \text{in case } m < n. \end{cases}$$

We make a collection of simple observations:

i Each x^γ is an element of I for any $\gamma < \alpha$ since $x_{m+1}^\gamma \leq \ell(x_m^\gamma)$ for any m;

ii We now see that xR_nx^γ for each $\gamma < \alpha$. For $m > n$ we obviously have that $x_m \geq x_m^\gamma$ and also $x_n > x_n^\gamma$ is clear. By induction we see that $x_m > x_m^\gamma$

using Lemma 5.8 and the fact that e is a strictly monotonously growing ordinal function;

iii $x_0 = y$ by definition;

iv By strict monotonicity of e, we see that for any $\gamma' < \gamma < \alpha$ we have $x^{\gamma} R_n x^{\gamma'}$.

□

We are now ready to prove Theorem 5.2.

Proof From Item (ii) to Item (iii) is easy and from Item (iii) to Item (i) is Lemma 5.9 so we focus on the remaining implication.

As mentioned before, half of the implication from Item (i) to Item (ii) follows from Corollary 5.7 so that it remains to show that $xR_n^{1+\alpha}y \Rightarrow xR_{n-1}^{e(1+\alpha)}y$. For $n = 0$ this is trivial and in case $n \neq 0$ we reason as follows.

Since $xR_n^{1+\alpha}y$ we get in particular that $x_n \geq y_n + 1 + \alpha$. Thus, by Lemma 5.8 we see

$$x_{n-1} \geq y_{n-1} + e(x_n) \geq y_{n-1} + e(y_n + 1 + \alpha).$$

Now using the fact (Lemma 5.5) that $e(y_n + 1 + \alpha) = e(y_n) \cdot e(1 + \alpha)$ we see, making a case distinction whether $y_n = 0$ or not and using that $e(1 + \alpha)$ is a limit ordinal, that

$$x_{n-1} \geq y_{n-1} + (1 + e(y_n) \cdot (1 + e(1 + \alpha)).$$

The result now follows from an application of Lemma 5.9. □

6 Definable sets

In this section we shall define a translation between formulas in MNF and Ignatiev sequences with finite support as well as a way of characterizing subsets of I. Moreover, we shall see how some of these subsets of I can be related to the extensions of formulas.

Definition 6.1 Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \ldots \wedge \langle n_k^{\alpha_k} \rangle \top \in \mathsf{MNF}$. By x_ψ we denote the sequence $\langle \pi_i(\psi) \rangle_{i<\omega}$.

In virtue of Definition 3.6, we can observe that for $\psi \in \mathsf{MNF}$, we have that $x_\psi \in \mathrm{Ig}^\omega$. Furthermore, we shall see that x_φ is the "first" point in I where φ holds. First we can make some simple observations.

Lemma 6.2

(i) *For any* $x \in I$, $x \Vdash \langle n^\alpha \rangle \top$ *iff* $x_n \geq \alpha$;

(ii) *For any* $\psi \in \mathsf{MNF}$, $x_\psi \Vdash \psi$.

Proof The second item follows from the first one and Definition 6.1. For the right-to-left implication of the first item, assume $x_n \geq \alpha > 0$. Therefore, for $i < n$, we have that $x_i > 0$ and for $i' > n$, $x_{i'} \geq 0$. Thus, by Theorem 5.2, $xR_n^\alpha \langle 0 \rangle$ and so $x \Vdash \langle n^\alpha \rangle \top$. For the other direction, assume $x \Vdash \langle n^\alpha \rangle \top$

for $\alpha > 0$. Hence, there is $y \in I$ such that $xR_n^\alpha y$ and $y \Vdash \top$. By Theorem 5.2, $x_n \geq y_n + \big(1 + e(y_{n+1})\big) \cdot \alpha$ and so, $x_n \geq \alpha$. The case $\alpha = 0$ is straightforward.$\Box$

The following two definitions introduce the extension of Ignatiev sequences and the extension of formulas, respectively.

Definition 6.3 Given $x \in I$, by $[\![x]\!]$ we denote the set of ℓ-sequences which are coordinate-wise at least as big as x. That is, we define $[\![x]\!] := \{y \in I : y_i \geq x_i \text{ for every } i < \omega\}$.

Definition 6.4 Let $\varphi \in \mathbb{F}$. By $[\![\, \varphi \,]\!]$ we denote the set of worlds where φ holds i.e. $[\![\, \varphi \,]\!] = \{x \in I : x \Vdash \varphi\}$.

The following lemma relates definitions 6.3 and 6.4.

Lemma 6.5 *For any $\varphi \in \mathbb{F}$, there is $x := \langle x_0, \ldots, x_k, 0\rangle \in I$ such that $[\![\, \varphi \,]\!] = [\![x]\!]$.*

Proof The proof goes by induction on φ. The base case is trivial. For **the conjunctive case**, let $\varphi = \psi \wedge \chi$. By the I.H. we have that there are $y, z \in I$ such that $[\![\, \psi \,]\!] = [\![y]\!]$ and $[\![\, \chi \,]\!] = [\![z]\!]$. Moreover, by the I.H. we also have that $y := \langle y_0, \ldots, y_j, 0\rangle$ and $z := \langle z_0, \ldots, z_i, 0\rangle$. Let n be the index of the rightmost non-zero component. Hence we can define x as follows:

- $x_i = \max(y_i, z_i)$ for $i \geq n$;
- $x_i = \min\{\delta : \delta \geq \max(y_i, z_i) \;\&\; l(\delta) \geq x_{i+1}\}$ for $i < n$.

We can easily check that $x \in I$. Next, we check that for any $x' \in I$, we have that $x' \Vdash \psi \wedge \chi$ iff $x' \in [\![x]\!]$. For right-to-left implication, consider $x' \in [\![x]\!]$. Thus, for $k < \omega$, we have that both $x'_k \geq x_k \geq y_k$ and $x'_k \geq x_k \geq z_k$. Thus, $x' \in [\![y]\!] \cap [\![z]\!]$ and so by the I.H. $x' \Vdash \psi \wedge \chi$. For the other direction, consider $x' \in I$ such that $x' \Vdash \psi \wedge \chi$. Clearly, for $i > n$, we have that $x'_i \geq x_i$. We check by induction on k that $x'_{n-k} \geq x_{n-k}$. For the base case, since $x' \Vdash \psi \wedge \chi$, then by the I.H. $x' \in [\![y]\!] \cap [\![z]\!]$ and so $x'_n \geq y_n$ and $x'_n \geq z_n$. Thus, $x'_n \geq \max(y_n, z_n) = x_n$. For the inductive step, by definition of Ignatiev sequences together with the I.H., we have that $l(x'_{n-(k+1)}) \geq x'_{n-k} \geq x_{n-k}$ and since $x' \Vdash \psi \wedge \chi$, then $x'_{n-(k+1)} \geq \max(y_{n-(k+1)}, z_{n-(k+1)})$. Therefore, being $x_{n-(k+1)}$ the minimal ordinal satisfying both conditions, we can conclude that $x'_{n-(k+1)} \geq x_{n-(k+1)}$. Hence, $[\![\, \psi \wedge \chi \,]\!] = [\![x]\!]$.

For **the modality case**, let $\varphi := \langle\, n^\alpha \,\rangle\psi$ with $\alpha > 0$. Thus, by the I.H. there is $y \in I$ such that $[\![\, \psi \,]\!] = [\![y]\!]$ and $y := \langle y_0, \ldots, y_j, 0\rangle$. We can define x as follows:

- $x_i = y_i$ for $i > n$;
- $x_n = y_n + \big(1 + e(y_{n+1})\big) \cdot \alpha$;
- $x_i = \min\{\delta : \delta \geq y_i \;\&\; l(\delta) \geq x_{i+1}\}$ for $i < n$.

As in the previous case, we can easily check that $x \in I$. We claim that $[\![x]\!] = [\![\langle\, n^\alpha \,\rangle\psi]\!]$. Let $x' \in [\![x]\!]$. By Theorem 5.2 we can see that $xR_n^\alpha y$. Hence, since $x'_i \geq x_i$ for $i < \omega$, $x' R_n^\alpha y$ and so $x' \Vdash \langle\, n^\alpha \,\rangle\psi$. For the other inclusion, consider

$x' \in I$ such that $x' \Vdash \langle n^\alpha \rangle\psi$. By the I.H. and Theorem 5.2, we can easily check that for $i > n$, we have that $x'_i \geq x_i$. For $i \leq n$, we proceed by an easy induction on k to see that $z_{n-k} \geq x_{n-k}$. The base case follows directly from Theorem 5.2. For the inductive step, by definition of Ignatiev sequences together with the I.H., we have that $l(x'_{n-(k+1)}) \geq x'_{n-k} \geq x_{n-k}$. Since $x' \Vdash \langle n^\alpha \rangle\psi$, then there is $z \in I$ such that $xR^\alpha_n z$ and $z \Vdash \psi$. Thus, by the I.H., $z \in [\![y]\!]$, and so we have that $x'_{n-(k+1)} > z_{n-(k+1)} \geq y_{n-(k+1)}$. Therefore, we get that $l(x'_{n-(k+1)}) \geq x_{n-k}$ and $x'_{n-(k+1)} > y_{n-(k+1)}$. Thus, since $x_{n-(k+1)}$ is the least ordinal satisfying both conditions, we have that $x'_{n-(k+1)} \geq x_{n-(k+1)}$. □

7 Soundness

To prove the soundness of **TSC**, let us begin by semantically define the entailment between our modal formulas.

Definition 7.1 For any formulas φ, $\psi \in \mathbb{F}$, we write $\varphi \models \psi$ iff for all $x \in I$, if $x \Vdash \varphi$ then $x \Vdash \psi$. Analogously, we write $\varphi \equiv_{\mathcal{J}} \psi$ iff for any $x \in I$, we have that $x \Vdash \varphi$ iff $x \Vdash \psi$.

With our notion of semantical entailment we can formulate our soundness theorem.

Theorem 7.2 (Soundness) *For any formulas $\varphi, \psi \in \mathbb{F}$, if $\varphi \vdash \psi$ then $\varphi \models \psi$.*

Proof By induction on the length of a **TSC** proof of $\varphi \vdash \psi$. It is easy to see that the first three rules preserve validity. With respect to the axioms, the first two axioms are easily seen to be valid. The the correctness of reduction axiom is given by Theorem 5.2. The remaining axioms and rules are separately proven to be sound in the remainder of this section. □

We start by proving the soundness of co-additivity axiom i.e.

$$\langle n^\alpha \rangle \langle n^\beta \rangle \varphi \equiv_{\mathcal{I}} \langle n^{\beta+\alpha} \rangle \varphi.$$

Proposition 7.3 *For any $x, z \in I$, $n < \omega$ and $\alpha, \beta < \Lambda$,*

$$\exists y \in I \ \left(xR^\alpha_n y \text{ and } yR^\beta_n z\right) \iff xR^{\beta+\alpha}_n z.$$

Proof We proceed by transfinite induction on α with the base case being trivial. For $\alpha \in \mathrm{Succ}$, let $\alpha := \delta + 1$ for some δ. Therefore:

$$\begin{aligned}
xR^\alpha_n y \text{ and } yR^\beta_n z \ &\Leftrightarrow\ xR^{\delta+1}_n y \text{ and } yR^\beta_n z;\\
&\Leftrightarrow\ \exists u \left(xR_n u \ \wedge\ uR^\delta_n y \ \wedge\ yR^\beta_n z\right);\\
&\Leftrightarrow\ \exists u \left(xR_n u \ \wedge\ uR^{\beta+\delta}_n z\right), \text{ by the I.H. };\\
&\Leftrightarrow\ xR^{\beta+\delta+1}_n z;\\
&\Leftrightarrow\ xR^{\beta+\alpha}_n z.
\end{aligned}$$

For $\alpha \in \text{Lim}$, we have that $xR_n^\alpha y$ and $yR_n^\beta z \;\Leftrightarrow\; \forall\delta < \alpha \left(xR_n^{1+\delta}y \;\wedge\; yR_n^\beta z\right)$ by Proposition 4.7. By the I.H. we obtain $\forall\delta < \alpha \; xR_n^{\beta+1+\delta}z$ and so $xR_n^{\beta+\alpha}z$. □

With this last result, we get the co-additivity of the R_n^α relations. This together with Definition 4.8 gives us the following corollary.

Corollary 7.4 *The co-additivity axiom is sound.*

Proof By Definition 4.8, $x \Vdash \langle n^\alpha \rangle\langle n^\beta \rangle\varphi$ iff there are $y, z \in I$ such that $xR_n^\alpha y$, $yR_n^\beta z$ and $z \Vdash \varphi$. Thus, by Proposition 7.3, $x \Vdash \langle n^\alpha \rangle\langle n^\beta \rangle\varphi$ iff $xR_n^{\beta+\alpha}z$ and $z \Vdash \varphi$ i.e. $x \Vdash \langle n^{\beta+\alpha} \rangle\varphi$. □

Proposition 7.5 *The monotonicity axiom is sound, that is:*

$$\langle n^\alpha \rangle\varphi \models \langle n^\beta \rangle\varphi$$

for $\beta < \alpha$.

Proof With the help Lemma 4.9, Item (iv), we have that if $x \Vdash \langle n^\alpha \rangle\varphi$ then $x \Vdash \langle n^\beta \rangle\varphi$ for β, $0 < \beta < \alpha$. We check that if $x \Vdash \langle n^1 \rangle\varphi$ then $x \Vdash \varphi$ by induction on φ.

The Base and the conjunctive cases are straightforward, so we consider $\varphi := \langle m^\delta \rangle\psi$ and assume $x \Vdash \langle n^1 \rangle\langle m^\delta \rangle\psi$. We make the following case distinction:

- If $n = m$, then by soundness of co-additivity axiom together with Lemma 4.9, Item (iv) we have that $x \Vdash \langle m^\delta \rangle\psi$;
- If $n > m$, then by monotonicity property of $R_n^{1+\alpha}$ together with soundness of co-additivity axiom and Lemma 4.9, Item (iv) we have that $x \Vdash \langle m^\delta \rangle\psi$;
- If $n < m$, then there are $y, z \in I$ such that $x\,R_n\,y\,R_m^\delta\,z$ and $z \Vdash \psi$. Thus, we can easily check that $x\,R_m^\delta\,z$, and so $x \Vdash \langle m^\delta \rangle\psi$.

□

The following proposition establishes the correction of the Schmerl axiom by using the translation between formulas in monomial normal form and Ignatiev sequences.

Proposition 7.6 *The Schmerl axiom is sound i.e.*

$$\langle n^\alpha \rangle\big(\langle n_0^{\alpha_0} \rangle\top \;\wedge\; \psi\big) \equiv_{\mathcal{J}} \langle n^{e^{n_0-n}(\alpha_0)\cdot(1+\alpha)} \rangle\top \wedge \langle n_0^{\alpha_0} \rangle\top \;\wedge\; \psi$$

for $n < n_0$ *and* $\langle n_0^{\alpha_0} \rangle\top \;\wedge\; \psi \in \mathsf{MNF}$.

Proof For the left-to-right direction, assume $x \Vdash \langle n^\alpha \rangle\big(\langle n_0^{\alpha_0} \rangle\top \;\wedge\; \psi\big)$. Thus, by soundness of monotonicity axiom, we have that $x \Vdash \langle n_0^{\alpha_0} \rangle\top \;\wedge\; \psi$. Therefore, we only need to check that $x \Vdash \langle n^{e^{n_0-n}(\alpha_0)\cdot(1+\alpha)} \rangle\top$. Notice that $x \Vdash \langle n^\alpha \rangle\langle n_0^{\alpha_0} \rangle\top$ and so there are $y, z \in I$ such that $xR_n^\alpha yR_{n_0}^{\alpha_0}z$. By Theorem 5.2 we have that

$$x_n \geq y_n + (1 + e(y_{n+1})) \cdot \alpha. \tag{1}$$

Also notice that since $yR_{n_0}^{\alpha_0}z$ then $yR_n^{e^{n_0-n}(\alpha_0)}z$ and $yR_{n+1}^{e^{n_0-n+1}(\alpha_0)}z$. Hence by Theorem 5.2 $y_n \geq e^{n_0-n}(\alpha_0)$ and $y_{n+1} \geq e^{n_0-n+1}(\alpha_0)$. Combining this with 1 we get that $x_n \geq e^{n_0-n}(\alpha_0) + \big(1 + e(e^{n_0-n+1}(\alpha_0))\big) \cdot \alpha = e^{n_0-n}(\alpha_0) \cdot (1+\alpha)$. Thus, in particular, we have that $xR_n^{e^{n_0-n}(\alpha_0)\cdot(1+\alpha)}\langle 0\rangle$ and so, $x \Vdash \langle n^{e^{n_0-n}(\alpha_0)\cdot(1+\alpha)}\rangle\top$.

For the other direction, assume $x \Vdash \langle n^{e^{n_0-n}(\alpha_0)\cdot(1+\alpha)}\rangle\top \wedge \langle n_0^{\alpha_0}\rangle\top \wedge \psi$. Hence, $x \Vdash \langle n^{e^{n_0-n}(\alpha_0)\cdot(1+\alpha)}\rangle\top$ and so, by Lemma 6.2, Item (i), $x_n \geq e^{n_0-n}(\alpha_0)\cdot(1+\alpha) = e^{n_0-n}(\alpha_0) + (1+e^{n_0-n}(\alpha_0))\cdot\alpha$. Since $\langle n_0^{\alpha_0}\rangle\top \wedge \psi \in \mathsf{MNF}$ consider $y_{\langle n_0^{\alpha_0}\rangle\top\wedge\psi}$. Notice that $\pi_n(\langle n_0^{\alpha_0}\rangle\top \wedge \psi) = e^{n_0-n}(\alpha_0)$, thus by Defintion 6.1 and Theorem 5.2 we can easly check that $xR_n^\alpha y_{\langle n_0^{\alpha_0}\rangle\top\wedge\psi}$ and by Lemma 6.2, Item (i), $y_{\langle n_0^{\alpha_0}\rangle\top\wedge\psi} \Vdash \langle n_0^{\alpha_0}\rangle\top\wedge\psi$. Therefore, $x \Vdash \langle n^\alpha\rangle\big(\langle n_0^{\alpha_0}\rangle\top \wedge \psi\big)$.□

Lastly, we check the soundness of Rule (iv) by applying the relation between definable sets and the extension of Ignatiev sequences proved in Lemma 6.5. This next result concludes the soundness proof of **TSC**.

Proposition 7.7 *If $\varphi \models \psi$ then, for $m < n$:*

$$\langle n^\alpha\rangle\varphi \wedge \langle m^{\beta+1}\rangle\psi \models \langle n^\alpha\rangle\big(\varphi \wedge \langle m^{\beta+1}\rangle\psi\big).$$

Proof Assume $\varphi \models \psi$ and let $x \in I$ such that $x \Vdash \langle n^\alpha\rangle\varphi \wedge \langle m^{\beta+1}\rangle\psi$. Since $\varphi \models \psi$, by Lemma 6.5, there are y, $z \in I$ such that $[\![y]\!] = [\![\varphi]\!] \subseteq [\![\psi]\!] = [\![z]\!]$. Let y', $z' \in I$ such that $[\![y']\!] = [\![\langle n^\alpha\rangle\varphi]\!]$ and $[\![z']\!] = [\![\langle m^{\beta+1}\rangle\psi]\!]$, and $w \in I$ such that $[\![w]\!] = [\![\varphi \wedge \langle m^{\beta+1}\rangle\psi]\!]$. Since $y \in [\![z]\!]$, we know that $w_i = y_i$ for $i > m$. For the remaining components, we have that:

- $w_m = \max\big(y_m, z'_m\big)$;
- $w_i = \min\{\delta : \delta \geq \max(y_i, z'_i)\ \&\ l(\delta) \geq w_{i+1}\}$ for $i < m$.

On the other hand, since $x \Vdash \langle n^\alpha\rangle\varphi \wedge \langle m^{\beta+1}\rangle\psi$, we have the following:

- $x_i \geq y_i$ for $i > n$;
- $x_n \geq y'_n$;
- $x_i \geq \min\{\delta : \delta \geq y'_i\ \&\ l(\delta) \geq x_{i+1}\}$ for $i,\ m < i < n$;
- $x_i \geq \min\{\delta : \delta \geq \max(y'_i, z'_i)\ \&\ l(\delta) \geq x_{i+1}\}$ for $i \leq m$.

It remains to be checked that $xR_n^\alpha w$. Clearly, $x_i \geq w_i$ for $i > n$. Also, since $w_n = y_n$, $w_{n+1} = y_{n+1}$ and $x_n \geq y'_n = y_n + \big(1 + e(y_{n+1})\big) \cdot \alpha$ we have that $x_n \geq w_n + \big(1 + e(w_{n+1})\big) \cdot \alpha$. Thus, we need to see that $x_i > w_i$ for $i < n$. For $i,\ m < i < n$, we can easily check that $y'_i > y_i = w_i$, and so $x_i > w_i$. For $i \leq m$, we show by induction on k that $x_{m-k} > w_{m-k}$. For the base case, we can have that $x_{m+1} > w_{m+1}$. Also we can observe that $\max(y'_m, z'_m) \geq \max(y_m, z'_m)$. Therefore $x_m > w_m$. For the inductive step, by the I.H. we have that $x_{m-k} > w_{m-k}$. Again, $\max(y'_{m-(k+1)}, z'_{m-(k+1)}) \geq \max(y_{m-(k+1)}, z'_{m-(k+1)})$, and so $x_{m-(k+1)} > w_{m-(k+1)}$. Hence, in virtue of Theorem 5.2 we get that $xR_n^\alpha w$, that is, $x \Vdash \langle n^\alpha\rangle\big(\varphi \wedge \langle m^{\beta+1}\rangle\psi\big)$. □

Although it is not needed later in this paper, we find it useful to observe that for any $x = \langle x_0, \ldots, x_k, 0 \rangle \in I$ there is $\psi \in \mathsf{MNF}$ so that $[\![x]\!] = [\![\psi]\!] = [\![x_\psi]\!]$. Having finite support is essential since e.g. the Ignatiev sequence $\langle \varepsilon_0, \varepsilon_0, \ldots \rangle \in I$ is not modally definable. To this regard, in [12] it is shown that a universal model for **TSC** can be built by just considering Ignatiev sequences with finite support.

8 Completeness

To establish the completeness of our system, first we need the following proposition that characterizes the non-derivability between formulas in monomial normal form.

Proposition 8.1 *Given $\varphi, \psi \in \mathsf{MNF}$, if $\varphi \nvdash \psi$ then there is $m_I \in \mathsf{N\text{-}mod}(\psi)$ such that $\pi_{m_I}(\varphi) < \pi_{m_I}(\psi)$.*

Proof This follows directly from Theorem 3.7 and Definition 3.6. □

Now we are ready to prove the completeness of **TSC**.

Theorem 8.2 (Completeness) *Given formulas $\varphi, \psi \in \mathbb{F}$, if $\varphi \models \psi$, then $\varphi \vdash \psi$.*

Proof By Theorem 3.5, w.l.o.g. let $\varphi, \psi \in \mathsf{MNF}$ such that $\varphi := \langle n_0^{\alpha_0} \rangle \top \wedge \ldots \wedge \langle n_k^{\alpha_k} \rangle \top$ and $\psi := \langle m_0^{\beta_0} \rangle \top \wedge \ldots \wedge \langle m_j^{\beta_j} \rangle \top$. Reasoning by contraposition, suppose $\varphi \nvdash \psi$. Therefore, by Proposition 8.1, we can conclude that for some $m_I \in \mathsf{N\text{-}mod}(\psi)$, we have that $\pi_{m_I}(\varphi) < \pi_{m_I}(\psi)$. Thus, consider the Ignatiev sequence x_φ. By Lemma 6.2, Item (ii), $x_\varphi \Vdash \varphi$ but $x_\varphi \nVdash \langle m_I^{\beta_I} \rangle \top$. Hence, $x_\varphi \nVdash \psi$ and so $\varphi \not\models \psi$. □

9 A calculus without normal forms

In this section we shall introduce a presentation of **TSC** that makes no use of formulas in monomial normal form. To this purpose, we shall introduce the notions of *increasing worms* and *m-β-ordinals*, and replace Schmerl's axiom by a new principle that establishes the derivability between these new formulas. The system obtained from this replacement is named **TSC***.

Definition 9.1 The set of *increasing worms*, denoted by IW is inductively defined as follows:

i) $\top \in \mathsf{IW}$;

ii) $\langle n^\alpha \rangle \top \in \mathsf{IW}$ for any $n < \omega$ and $\alpha,\ 0 < \alpha < \Lambda$;

iii) if $\langle n^\alpha \rangle A \in \mathsf{IW}$ and $m < n$, then $\langle m^\beta \rangle \langle n^\alpha \rangle A \in \mathsf{IW}$.

Definition 9.2 Let $\langle n^\alpha \rangle A \in \mathsf{IW}$, $m < n$ and $\beta < \Lambda$. By $o_m^\beta(\langle n^\alpha \rangle A)$ we denote the *m-β-ordinal* of $\langle n^\alpha \rangle A$, that is recursively defined as follows:

i) $o_m^\beta(\langle n^\alpha \rangle \top) = e^{n-m}(\alpha) \cdot (1 + \beta)$;

ii) $o_m^\beta(\langle n^\alpha \rangle A) = e^{n-m}(o_n^\alpha(A)) \cdot (1 + \beta)$.

For any $m < \omega$ and $\beta < \Lambda$, we set $o_m^\beta(\top)$ to be zero.

By **TSC*** we denote the system obtained by substituting in **TSC** the Schmerl axiom by the following principle:

$$\text{IW axioms: } \langle n^{\alpha} \rangle A \equiv \langle n^{o^{\alpha}_{n}(A)} \rangle \top \ \wedge \ A$$

for $\langle n^{\alpha} \rangle A \in \mathsf{IW}$.

9.1 MNF's and IW's

To prove the equivalence between both systems, first we shall see how formulas in monomial normal form and increasing worms are related. Therefore, in the following lemmata we state how every formula in monomial normal form is equivalent to an increasing worm, modulo **TSC***, and likewise, that every increasing worm is **TSC**-equivalent to a formula in monomial normal form.

From now on we will use the following notation: given $\varphi,\ \psi \in \mathbb{F}$, we write $\varphi \vdash_{\mathbf{TSC}} \psi$ ($\varphi \vdash_{\mathbf{TSC}^*} \psi$) to denote that the sequent $\varphi \vdash \psi$ is derivable in **TSC** (**TSC***). Analogously, we use $\varphi \equiv_{\mathbf{TSC}} \psi$ ($\varphi \equiv_{\mathbf{TSC}^*} \psi$) to denote that both $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are derivable in **TSC** (**TSC***).

Lemma 9.3 *For every* $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \ldots \wedge \langle n_k^{\alpha_k} \rangle \top \in \mathsf{MNF}$ *there is an* $A \in \mathsf{IW}$ *such that:*

(i) $A \equiv_{\boldsymbol{TSC}^*} \psi$;

(ii) $A := \langle n_0^{\beta_0} \rangle \ldots \langle n_k^{\beta_k} \rangle \top$ *where:*
  (a) $\alpha_k = \beta_k$ *and*
  (b) $\alpha_i = o^{\beta_i}_{n_i}\big(\langle n_{i+1}^{\beta_{i+1}} \rangle \ldots \langle n_k^{\beta_k} \rangle \top \big)$ *for* $i,\ 0 \leq i < k$.

Proof By induction on k. The base case is trivial and the inductive case follows from the I.H. and the IW axiom. □

Lemma 9.4 *For any* $A := \langle n_0^{\beta_0} \rangle \ldots \langle n_k^{\beta_k} \rangle \top \in \mathsf{IW}$ *there is a unique* $\psi \in \mathsf{MNF}$ *such that:*

(i) $\psi \equiv_{\boldsymbol{TSC}} A$;

(ii) $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \ldots \wedge \langle n_k^{\alpha_k} \rangle \top$ *where:*
  (a) $\beta_k = \alpha_k$ *and*
  (b) $\alpha_i = o^{\beta_i}_{n_i}\big(\langle n_{i+1}^{\beta_{i+1}} \rangle \ldots \langle n_k^{\beta_k} \rangle \top \big)$ *for* $i,\ 0 \leq i < k$.

Proof By induction k. The base case is straightforward. The inductive step follows from the Schmerl axiom together with the I.H. □

9.2 Equivalence between TSC* and TSC

With these results established in the previous subsection, we are ready to prove the equivalence of both systems by checking the interderivability of Schmerl and IW axioms.

Proposition 9.5 *The Schmerl axiom is derivable in* ***TSC**** *i.e.*

$$\langle n^{\alpha} \rangle \big(\langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi \big) \equiv_{\boldsymbol{TSC}^*} \langle n^{e^{n_0 - n}(\alpha_0)\cdot(1+\alpha)} \rangle \top \wedge \langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi$$

for $n < n_0$ *and* $\langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi \in \mathsf{MNF}$.

Proof By Lemma 9.3 we have that:

$$\langle n^{\alpha} \rangle \big(\langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi \big) \equiv_{\mathbf{TSC}^*} \langle n^{\alpha} \rangle \langle n_0^{\beta_0} \rangle A$$

with $\langle n_0^{\alpha^0} \rangle \top \wedge \psi \equiv_{\mathbf{TSC}^*} \langle n_0^{\beta_0} \rangle A \in \mathsf{IW}$. Thus, by the IW axiom, we get that

$$\langle n^{\alpha} \rangle \big(\langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi \big) \equiv_{\mathbf{TSC}^*} \langle n^{o_n^{\alpha}\left(\langle n_0^{\beta_0} \rangle A\right)} \rangle \top \ \wedge \ \langle n_0^{\beta_0} \rangle A,$$

and so

$$\langle n^{\alpha} \rangle \big(\langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi \big) \equiv_{\mathbf{TSC}^*} \langle n^{o_n^{\alpha}\left(\langle n_0^{\beta_0} \rangle A\right)} \rangle \top \ \wedge \ \langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi.$$

Since $o_n^{\alpha}\big(\langle n_0^{\beta_0} \rangle A\big) = e^{n_0 - n}\big(o_{n_0}^{\beta_0}(A) \big) \cdot (1+\alpha)$ and by Lemma 9.3, $o_{n_0}^{\beta_0}(A) = \alpha_0$, we can conclude that $o_n^{\alpha}\big(\langle n_0^{\beta_0} \rangle A\big) = e^{n_0 - n}(\alpha_0) \cdot (1+\alpha)$ and therefore:

$$\langle n^{\alpha} \rangle \big(\langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi \big) \equiv_{\mathbf{TSC}^*} \langle n^{e^{n_0-n}(\alpha_0)\cdot(1+\alpha)} \rangle \top \ \wedge \ \langle n_0^{\alpha_0} \rangle \top \ \wedge \ \psi.$$

□

Proposition 9.6 *The IW axiom is derivable in* ***TSC*** *i.e.*

$$\langle n^{\alpha} \rangle A \equiv_{\boldsymbol{TSC}} \langle n^{o_n^{\alpha}(A)} \rangle \top \ \wedge \ A$$

for $\langle n^{\alpha} \rangle A \in \mathsf{IW}$.

Proof By induction on A with base case being trivial. For the inductive step, let $A := \langle n_0^{\beta_0} \rangle A'$. By Lemma 9.4, we have that

$$\langle n^{\alpha} \rangle A \equiv_{\mathbf{TSC}} \langle n^{\alpha} \rangle \big(\langle n_0^{o_{n_0}^{\beta_0}(A')} \rangle \top \ \wedge \ \psi \big)$$

for $\langle n_0^{\beta_0} \rangle A' \equiv_{\mathbf{TSC}} \langle n_0^{o_{n_0}^{\beta_0}(A')} \rangle \top \wedge \psi \in \mathsf{MNF}$, and so by Schmerl's axiom:

$$\langle n^{\alpha} \rangle A \equiv_{\mathbf{TSC}} \langle n^{e^{n_0-n}\left(o_{n_0}^{\beta_0}(A')\right)\cdot(1+\alpha)} \rangle \ \wedge \ \langle n_0^{o_{n_0}^{\beta_0}(A')} \rangle \top \ \wedge \ \psi.$$

Thus,

$$\langle n^{\alpha} \rangle A \equiv_{\mathbf{TSC}} \langle n^{o_n^{\alpha}\left(\langle n_0^{\beta_0} \rangle A'\right)} \rangle \ \wedge \ \langle n_0^{o_{n_0}^{\beta_0}(A')} \rangle \top \ \wedge \ \psi,$$

that is, $\langle n^{\alpha} \rangle A \equiv_{\mathbf{TSC}} \langle n^{o_n^{\alpha}\left(\langle n_0^{\beta_0} \rangle A'\right)} \rangle \ \wedge \ A$. □

Corollary 9.7 *For any* $\varphi, \ \psi \in \mathbb{F}$,

$$\varphi \vdash_{\boldsymbol{TSC}} \psi \iff \varphi \vdash_{\boldsymbol{TSC}^*} \psi.$$

Proof By induction on the length of the proof. It follows immediately from Propositions 9.5 and 9.6. □

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