A canonical model construction for intuitionistic distributed knowledge

Gerhard Jäger

Institute of Computer Science, University of Bern
Neubrückstrasse 10, CH-3012 Bern, Switzerland
jaeger@inf.unibe.ch

Michel Marti [1]

Institute of Computer Science, University of Bern
Neubrückstrasse 10, CH-3012 Bern, Switzerland
mmarti@inf.unibe.ch

Abstract

Intuitionistic epistemic logic is an active research field. However, so far no consensus has been reached what the correct form of intuitionistic epistemic logic is and more technical and conceptual work is needed to obtain a better understanding. This article tries to make a small technical contribution to this enterprise.
Roughly speaking, a proposition is distributed knowledge among a group of agents if it follows from their combined knowledge. We are interested in formalizing intuitionistic distributed knowledge. Our focus is on two theories IDK and IDT, presented as Hilbert-style systems, and the proof of the completeness of these theories; their correctness is obvious.
Intuitionistic distributed knowledge is semantically treated following the standard lines of intuitionistic modal logic. Motivated by an approach due to Fagin, Halpern, and Vardi, though significantly simplified for the treatment of IDK and IDT, we show completeness of these systems via a canonical model construction.

Keywords: Distributed knowledge, intuitionistic modal logic, canonical models.

1 Introduction

Intuitionistic epistemic logic is an active research field; see, for example, Artemov and Protopopescu [1], Hirai [6], Jäger and Marti [7], Krupski and Yatmanov [8], Proietti [11], Suzuki [14] and also the somewhat older Williamson [16]. The two main pillars of most present approaches are:

- Epistemic logic based on classical modal logic. There exists a huge amount of work making case for classical multi-modal systems providing an adequate

[1] Research partly supported by the Swiss National Science Foundation.

and useful framework for reasoning about knowledge and belief. The textbooks Fagin, Halpern, Moses, and Vardi [2] and Meyer and van der Hoek [9] provide a solid introduction into this area.

- Systems of intuitionistic modal logic. There is also the interesting – though not so popular – world of intuitionistic modal logic. A fundamental result is the completeness proof for the logic **IK** in Fischer Servi [12], and Simpson [13] provides an excellent survey of intuitionistic modal logics, contains some further results and leads to present research in this area.

However, so far no consensus has been reached what the "correct" form of intuitionistic epistemic logic is. Different approaches have been proposed, varying in their philosophical justifications and taking into account various fields of possible applications. We believe that intuitionistic epistemic logic may provide an approach to dealing with knowledge that is more "constructive" than the treatment of epistemic logic based on classical logic. In particular, it has to be seen whether the process of building up (or acquiring) knowledge by an agent can be more naturally formalized in an intuitionistic environment. It is clear that more technical and conceptual work is needed and that we have to develop a better understanding of the general methodology behind intuitionistic epistemic reasoning.

This article tries to make a small technical contribution to this enterprise. It can be considered as a twin of Jäger and Marti [7] which deals with intuitionistic common knowledge. Now we are interested in formalizing intuitionistic distributed knowledge. Our focus is on two theories **IDK** and **IDT** and the proof of the completeness of these theories. This is achieved by adapting a canonical model construction for our framework.

2 The language $\mathcal{L}_{DK}$ and its semantics

Our general scenario is that we want to deal with ℓ agents $ag_1, \ldots, ag_\ell$, the individual knowledge/belief of these agents and knowledge/belief distributed among them. In order to avoid a trivial situation, $\ell \geq 2$ is a general assumption. We begin with introducing a language $\mathcal{L}_{DK}$ tailored for this purpose and interpret its formulas over so-called epistemic Kripke structures, thus providing a semantic approach to intuitionistic distributed knowledge/belief. To formally express that agent ag_i knows or believes α, we will write $\mathsf{K}_i(\alpha)$, whereas $\mathsf{D}(\alpha)$ says that α is knowledge distributed among $ag_1, \ldots, ag_\ell$. Hence the language $\mathcal{L}_{DK}$ comprises the following primitive symbols:

PS.1 Countably many atomic propositions p, q, r (possibly with subscripts); the collection of all atomic propositions is called $PROP$.

PS.2 The logical constant $\bot$ and the logical connectives $\vee$, $\wedge$, and $\rightarrow$.

PS.3 The modal operators $\mathsf{K}_1, \ldots, \mathsf{K}_\ell, \mathsf{D}$.

The *formulas* $\alpha, \beta, \gamma, \delta$ (possibly with subscripts) of $\mathcal{L}_{DK}$ are generated by the following BNF:

$$\alpha \ ::\equiv \ p \ \mid \ \bot \ \mid \ (\alpha \vee \alpha) \ \mid \ (\alpha \wedge \alpha) \ \mid \ (\alpha \to \alpha) \ \mid \ \mathsf{K}_i(\alpha) \ \mid \ \mathsf{D}(\alpha).$$

It is common in intuitionistic logic to define negation $\neg\alpha$ by $(\alpha \to \bot)$ and equivalence $(\alpha \leftrightarrow \beta)$ by $((\alpha \to \beta) \wedge (\beta \to \alpha))$. We often omit parentheses and brackets if there is no danger of confusion.

As in the classical setting all operators K_i will have the normality axiom

$$\mathsf{K}_i(\alpha \to \beta) \to (\mathsf{K}_i(\alpha) \to \mathsf{K}_i(\beta)).$$

Sometimes it is argued that interpreting $\mathsf{K}_i(\alpha)$ as "agent ag_i knows α" requires the presence of the truth property

$$\mathsf{K}_i(\alpha) \to \alpha$$

and possibly positive as well as negative introspection; otherwise $\mathsf{K}_i(\alpha)$ should be seen as stating that agent ag_i only believes α. However, since we are primarily interested in technical questions, we do not make this distinction and speak of knowledge and distributed knowledge to simplify matters.

In this paper we do not enter into a discussion of the modal logic approach to knowledge and distributed knowledge. As mentioned above, this is done in great detail in the textbooks Fagin, Halpern, Moses, and Vardi [2] and Meyer and van der Hoek [9] as well as in many research articles; see, e.g., Fagin, Halpern, and Vardi [3], Gerbrandy [4], Hakli and Negri [5], and Wang and Ågotnes [15]. However, all these texts are about distributed knowledge based on classical logic, whereas here we work in the context of intuitionistic logic. As mentioned above, intuitionistic epistemic logic is interesting by its own. Here we add the facet of intuitionistic distributed knowledge/belief to the general discussion.

First we have to fix the adequate structures over which the formulas of $\mathcal{L}_{DK}$ will be interpreted. First some notation: Given a non-empty set W and a binary relation R on W, we often write aRb for $(a, b) \in R$ and set $R[a] := \{b \in W : aRb\}$. We say that $R[a]$ is the collection of all elements of W that are accessible from a via R.

Definition 2.1 An *epistemic Kripke structure (EK-structure for short) of order* ℓ is an $(\ell + 3)$-tuple $\mathfrak{M} = (W, \preceq, R_1, \ldots, R_\ell, V)$ with the following properties:

(EK.1) W is a nonempty set (the set of the so-called worlds of $\mathfrak{M}$) and $\preceq$ is a preorder on W.

(EK.2) Every R_i for $1 \leq i \leq \ell$ is a binary relation on W such that for any $a, b \in W$,

$$a \preceq b \quad \Longrightarrow \quad R_i[b] \subseteq R_i[a].$$

(EK.3) V is a function from W to the power set of $PROP$ such that for any $a, b \in W$,

$$a \preceq b \quad \Longrightarrow \quad V(a) \subseteq V(b).$$

$\mathfrak{M}$ is called *a reflexive EK-structure* iff all relations $R_1, \ldots, R_\ell$ are reflexive.

(EK.1) and (EK.3) are the usual properties of a Kripke structure for intuitionistic propositional logic. Given the EK-structure of order ℓ

$$\mathfrak{M} = (W, \preceq, R_1, \ldots, R_\ell, V),$$

the relation R_i is the accessibility relation associated with agent ag_i that tells which worlds b are accessible for ag_i from world a; agent ag_i "knows" α in world a iff α holds in all worlds that are accessible for ag_i from world a. On the other hand, α is considered to be distributed knowledge in world a iff α holds in those worlds that are accessible for <u>all</u> agents $ag_1, \ldots, ag_\ell$ from a. The condition (EK.2) ensures monotonicity for formulas of the form $\mathsf{K}_i(\alpha)$. Whenever agent ag_i progresses along $\preceq$, the collection of worlds that are accessible for ag_i can go down, reflecting the fact that some worlds are ruled out as being accessible due to new information. If R_i is reflexive then all worlds b such that $a \preceq b$ are accessible for agent ag_i from a.

Definition 2.2 [Value] Given an EK-structure $\mathfrak{M} = (W, \preceq, R_1, \ldots, R_\ell, V)$ of order ℓ, the set $\|\alpha\|_{\mathfrak{M}}$ of worlds satisfying α is inductively defined as follows:

(1) $\|\bot\|_{\mathfrak{M}} := \emptyset$,

(2) $\|p\|_{\mathfrak{M}} := \{a \in W : p \in V(a)\}$ for any $p \in PROP$,

(3) $\|\alpha \vee \beta\|_{\mathfrak{M}} := \|\alpha\|_{\mathfrak{M}} \cup \|\beta\|_{\mathfrak{M}}$,

(4) $\|\alpha \wedge \beta\|_{\mathfrak{M}} := \|\alpha\|_{\mathfrak{M}} \cap \|\beta\|_{\mathfrak{M}}$,

(5) $\|\alpha \to \beta\|_{\mathfrak{M}} := \{a \in W : \{b \in W : a \preceq b\} \cap \|\alpha\|_{\mathfrak{M}} \subseteq \|\beta\|_{\mathfrak{M}}\}$,

(6) $\|\mathsf{K}_i(\alpha)\|_{\mathfrak{M}} := \{a \in W : R_i[a] \subseteq \|\alpha\|_{\mathfrak{M}}\}$,

(7) $\|\mathsf{D}(\alpha)\|_{\mathfrak{M}} := \{a \in W : \bigcap_{i=1}^{\ell} R_i[a] \subseteq \|\alpha\|_{\mathfrak{M}}\}$.

A simple proof by induction on the structure of α shows that the sets $\|\alpha\|_{\mathfrak{M}}$ satisfy the usual monotonicity condition of intuitionistic logic.

Lemma 2.3 *For all EK-structures $\mathfrak{M} = (W, \preceq, R_1, \ldots, R_\ell, V)$ of order ℓ, all elements $a, b \in W$, and all α we have that*

$$a \preceq b \ \textit{and} \ a \in \|\alpha\|_{\mathfrak{M}} \quad \Longrightarrow \quad b \in \|\alpha\|_{\mathfrak{M}}.$$

We call $\|\alpha\|_{\mathfrak{M}}$ the *value* of α in $\mathfrak{M}$ and often write $(\mathfrak{M}, a) \models \alpha$ instead of $a \in \|\alpha\|_{\mathfrak{M}}$. In addition, α is *valid in the EK-structure* $\mathfrak{M}$, written $\mathfrak{M} \models \alpha$, iff $(\mathfrak{M}, a) \models \alpha$ for all worlds a of $\mathfrak{M}$. Finally, α is called *EK-valid*, written $\models \alpha$, iff α is valid in every EK-structure. Analogously, α is called *reflexive EK-valid*, written $\models_{ref} \alpha$, iff α is valid in every reflexive EK-structure.

We end this section with comparing our semantics to some common approaches in the literature, in particular that of Fischer Servi, Plotkin and Sterling, and Simpson. Since distributed knowledge is not treated there, we confine ourselves to D-free $\mathcal{L}_{DK}$ formulas for this comparison. The mentioned authors impose certain restrictions on their frames to deal with the interplay between $\Box$- and $\Diamond$-formulas. Our modal operators K_i are boxes, as is the operator D for distributed knowledge. This means that we work in multi-agent versions of the $\Box$-fragment, and therefore do not need these frame conditions.

Intuitionistic logic requires monotonicity, and in Fischer Servi [12], Plotkin and Sterling [10], and Simpson [13] this is done by building it into the truth definition. As shown in Jäger and Marti [7], both approaches lead to equivalent notions of validity. So our semantics for intuitionistic distributed knowledge builds on established semantic concepts.

The question now is whether there exist deductive systems that prove exactly the EK-valid and reflexive EK-valid formulas, respectively.

3 The Hilbert systems IDK and IDT

In the following we present a Hilbert-style axiomatization **IDK** of intuitionistic distributed knowledge and the system **IDT** for intuitionistic distributed knowledge with the truth property. There are also natural sequent calculi that prove the same formulas, but for the model construction and completeness proofs below it is irrelevant what kind of deductive system we use.

The **axioms of IDK** comprise the usual axioms of intuitionistic propositional logic, the normality axioms, sometimes also called K-axioms,

$$\mathsf{K}_i(\alpha \to \beta) \;\to\; (\mathsf{K}_i(\alpha) \to \mathsf{K}_i(\beta)) \tag{\textbf{K}}$$

plus the D-axioms

$$\mathsf{D}(\alpha \to \beta) \;\to\; (\mathsf{D}(\alpha) \to \mathsf{D}(\beta)), \tag{\textbf{D1}}$$

$$\mathsf{K}_i(\alpha) \to \mathsf{D}(\alpha), \tag{\textbf{D2}}$$

always for all i with $1 \leq i \leq \ell$ and all α, β. Because of (**D1**) the operator D is normal, and in view of (**D2**) anything known by any agent is distributed knowledge.

The rules of inference of **IDK** are modus ponens and necessitation for the operators $\mathsf{K}_1, \ldots, \mathsf{K}_\ell$ and all α, β:

$$\frac{\alpha \qquad \alpha \to \beta}{\beta}\ (\textbf{MP}) \qquad \text{and} \qquad \frac{\alpha}{\mathsf{K}_i(\alpha)}\ (\textbf{NEC}).$$

Because of (**D2**) and (**NEC**) the necessitation rule for D

$$\frac{\alpha}{\mathsf{D}(\alpha)}$$

is derivable in **IDK**. It is easy to see that all axioms of **IDK** are EK-valid; (**D1**) and (**D2**) follow directly from the intersection-interpretation of D. Furthermore, all EK-structures are clearly closed under the rules of inference of **IDK**.

The theory **IDT** is obtained from **IDK** by adding the claim that the distributed knowledge of α implies α, i.e.,

$$\mathsf{D}(\alpha) \to \alpha \tag{\textbf{T}}$$

for all α. In view of (**D2**) this implies the truth property $\mathsf{K}_i(\alpha) \to \alpha$ for all operators K_i.

Those EK-structures of order ℓ in which all (**T**)-axioms are valid are called *(**T**)-models*. The intended structures for **IDT** are reflexive EK-structures, and (**T**) is obviously valid in those. However, there are non-reflexive EK-structures of order ℓ in which all (**T**)-axioms are valid.

Let **ID•** be one of the theories **IDK** or **IDT**. We write **ID•** $\vdash \alpha$ to state that α is provable in the theory **ID•** in the usual sense. The following soundness theorem is then straightforwardly proved by induction on the length of the derivations.

Theorem 3.1 (Soundness) *For all α we have:*

(i) **IDK** $\vdash \alpha \quad \Longrightarrow \quad \models \alpha$.

(ii) **IDT** $\vdash \alpha \quad \Longrightarrow \quad \models_{ref} \alpha$.

As mentioned above, there exist non-reflexive (**T**)-models. Nevertheless, validity of (**T**) in EK-structures is closely related to reflexivity, as show in the following lemma. First a definition.

Definition 3.2 Let $\mathfrak{M} = (W, \preceq, R_1, \dots, R_\ell, V)$ be an EK-structure of order ℓ. The *reflexive extension* $\overline{\mathfrak{M}}$ of $\mathfrak{M}$ is defined to be the structure

$$(W, \preceq \overline{R}_1, \dots, \overline{R}_\ell, V),$$

where (for $1 \leq i \leq \ell$) the relation $\overline{R}_i$ is defined to be the reflexive closure of R_i, i.e., $\overline{R}_i := R_i \cup \{(a, a) : a \in W\}$.

It is an easy observation that any (**T**)-model can be extendend to a reflexive EK-structure of the same order.

Lemma 3.3 *If $\mathfrak{M} = (W, \preceq, R_1, \dots, R_\ell, V)$ is a (**T**)-model, then $\overline{\mathfrak{M}}$ is a reflexive EK-structure; for all $a \in W$ and all α we have that*

$$(\overline{\mathfrak{M}}, a) \models \alpha \quad \Longleftrightarrow \quad (\mathfrak{M}, a) \models \alpha.$$

Proof. The reflexivity of $\overline{\mathfrak{M}}$ is clear. The second part is proved by induction on α. If α is the logical constant $\bot$ or an atomic proposition, the assertion is obvious; if α is a disjunction, a conjunction, or an implication it follows directly from the induction hypothesis. Hence we can concentrate on the cases that α is of the form $\mathsf{K}_i(\beta)$ or $\mathsf{D}(\beta)$.

(i) Let α be the formula $\mathsf{K}_i(\beta)$. The direction from left to right is evident. So asssume $(\mathfrak{M}, a) \models \mathsf{K}_i(\beta)$, from which we obtain

$$(\mathfrak{M}, b) \models \beta \quad \text{for all } b \in R_i[a].$$

$\mathfrak{M}$ is a (**T**)-model, thus $\mathsf{K}_i(\beta) \to \beta$ is valid in $\mathfrak{M}$ and our assumption also yields $(\mathfrak{M}, a) \models \beta$. From the induction hypothesis we obtain $(\overline{\mathfrak{M}}, c) \models \beta$ for all $c \in R_i[a] \cup \{a\} = \overline{R}_i[a]$. Therefore, $(\overline{\mathfrak{M}}, a) \models \mathsf{K}_i(\beta)$.

(ii) Let α be the formula $\mathsf{D}(\beta)$. The direction from left to right is evident again. To show the converse direction, let $(\mathfrak{M}, a) \models \mathsf{D}(\beta)$. Hence we have

$$(\mathfrak{M}, b) \models \beta \quad \text{for all } b \in \bigcap_{i=1}^{\ell} R_i[a].$$

Since $\mathfrak{M}$ is a (**T**)-model, we also have $(\mathfrak{M}, a) \models \beta$. Hence the induction hypothesis implies $(\overline{\mathfrak{M}}, c) \models \beta$ for all $c \in \bigcap_{i=1}^{\ell} R_i[a] \cup \{a\} = \bigcap_{i=1}^{\ell} \overline{R}_i[a]$. This is what we had to show. ☐

4 Pseudo-validity

Now we build up some machinery that will lead to the canonical models and the completeness proofs for the systems **IDK** and **IDT** in the next section. What we do here is motivated by the approach presented in Fagin, Halpern, and Vardi [3] and Wang and Ågotnes [15]. However, our version is a significant simplification, tailored for the treatment of **IDK** and **IDT**.

The idea is to introduce the notion of pseudo-validity. In doing that, we interpret the formulas in EK-structures of order $(\ell + 1)$ where the operator D is interpreted by the additional binary accessibility relation $R_{\ell+1}$. Afterwards we will extend these EK-structures of order $(\ell + 1)$ to strict EK-structures of order $(\ell + 1)$ and then collapse these strict EK-structures of order $(\ell + 1)$ to EK-structures of order ℓ, suitable for our purpose.

Definition 4.1 Given an EK-structure $\mathfrak{M} = (W, \preceq, R_1, \ldots, R_{\ell+1}, V)$ of order $(\ell + 1)$, the set $\|\alpha\|^{ps}_{\mathfrak{M}}$ of worlds pseudo-satisfying α is inductively defined as follows: Clauses (1) to (6) are as in Definition 2.2, but clause (7) is replaced by

(7') $\|\mathsf{D}(\alpha)\|^{ps}_{\mathfrak{M}} := \{a \in W : R_{\ell+1}[a] \subseteq \|\alpha\|^{ps}_{\mathfrak{M}}\}$.

As can be seen by a trivial induction on α also this assignment of sets of worlds to formulas is monotone.

Lemma 4.2 *For every EK-structure $\mathfrak{M} = (W, \preceq, R_1, \ldots, R_{\ell+1}, V)$ of order $(\ell + 1)$, all elements $a, b \in W$, and all α we have that*

$$a \preceq b \text{ and } a \in \|\alpha\|^{ps}_{\mathfrak{M}} \quad \Longrightarrow \quad b \in \|\alpha\|^{ps}_{\mathfrak{M}}.$$

We call $\|\alpha\|^{ps}_{\mathfrak{M}}$ the *pseudo-value* of α in $\mathfrak{M}$ and often write $(\mathfrak{M}, a) \models^{ps} \alpha$ instead of $a \in \|\alpha\|^{ps}_{\mathfrak{M}}$. In addition, α is *pseudo-valid in the EK-structure* $\mathfrak{M}$ of order $(\ell + 1)$, written $\mathfrak{M} \models^{ps} \alpha$, iff $(\mathfrak{M}, a) \models^{ps} \alpha$ for all worlds a of $\mathfrak{M}$.

Since the operator D is interpreted by the relation $R_{\ell+1}$ in an EK-structure $\mathfrak{M} = (W, \preceq, R_1, \ldots, R_{\ell+1}, V)$ of order $(\ell+1)$, the axioms (**D2**) are not necessarily pseudo-valid in $\mathfrak{M}$. Those EK-structures of order $(\ell + 1)$ in which all (**D2**)-axioms are pseudo-valid are called *(**D2**)-pseudo-models*. An EK-structure $\mathfrak{M}$ of order $(\ell + 1)$ is a *(**D2T**)-pseudo-model* iff all (**D2**)-axioms and all (**T**)-axioms are pseudo-valid in $\mathfrak{M}$.

Of course, for every EK-structure $\mathfrak{M}$ of order ℓ there is an EK-structure $\mathfrak{M}'$ of order $(\ell+1)$ such that validity in $\mathfrak{M}$ is equivalent to pseudo-validity in $\mathfrak{M}'$. The following lemma is an immediate consequence of Definition 2.2 and Definition 4.1.

Lemma 4.3 *Let $\mathfrak{M} = (W, \preceq, R_1, \ldots, R_\ell, V)$ be an EK-structure of order ℓ and define*

$$\mathfrak{M}' := (W, \preceq, R_1, \ldots, R_\ell, \bigcap_{i=1}^{\ell} R_i, V).$$

Then $\mathfrak{M}'$ is an EK-structure of order $(\ell+1)$ and for all $a \in W$ and all α we have that

$$(\mathfrak{M}, a) \models \alpha \iff (\mathfrak{M}', a) \models^{ps} \alpha.$$

*In particular, $\mathfrak{M}'$ is a (**D2**)-pseudo-model and if $\mathfrak{M}$ is a (**T**)-model, then $\mathfrak{M}'$ is a (**D2T**)-pseudo-model.*

Our next step is to transform a given EK-structure of order $(\ell+1)$ into what we call its strict extension. The purpose of this extension is to enforce a well-controlled behavior of the intersection of the accessibility relations. From now on we write I for the set $\{1, \ldots, (\ell+1)\}$.

Definition 4.4 [Strict extension] Given an EK-structure

$$\mathfrak{M} = (W, \preceq, R_1, \ldots, R_{\ell+1}, V)$$

of order $(\ell+1)$, its *strict extension* is defined to be the structure

$$\mathfrak{M}^\sharp = (W^\sharp, \preceq^\sharp, R_1^\sharp, \ldots, R_{\ell+1}^\sharp, V^\sharp),$$

where we set:

($\sharp$1) $W^\sharp := W \times I$,

($\sharp$2) $\preceq^\sharp := \{((a,i),(b,j)) : a \preceq b \text{ and } i, j \in I\}$,

($\sharp$3) $R_i^\sharp := \{((a,j),(b,i)) : (a,b) \in R_i \text{ and } j \in I\}$ for any $i \in I$,

($\sharp$4) $V^\sharp((a,i)) := V(a)$ for any $(a,i) \in W^\sharp$.

It is obvious that $\mathfrak{M}^\sharp$ is an EK-structure of order $(\ell+1)$. Further properties of strict extensions are summarized in the following lemma whose proof is obvious.

Lemma 4.5 *Let $\mathfrak{M} = (W, \preceq, R_1, \ldots, R_{\ell+1}, V)$ be an EK-structure of order $(\ell+1)$. Then we have:*

(i) *If i and j are different elements of I, then $R_i^\sharp[(a,k)] \cap R_j^\sharp[(a,k)] = \emptyset$ for any $(a,k) \in W^\sharp$.*

(ii) $\bigcap_{i=1}^{\ell} R_i^\sharp[(a,k)] = \emptyset$ *for any $(a,k) \in W^\sharp$.*

Proof. The second assertion is an immediate consequence of the first since we deal with at least two agents. The first assertion follows from ($\sharp$3), which claims that all elements of $W^\sharp$ accessible from (a,k) via $R_i^\sharp$ are of the form (b,i) and those accessible from (a,k) via $R_j^\sharp$ are of the form (c,j). □

The following lemma is important and shows that the strict extension of an EK-structure of order $(\ell+1)$ does not affect the class of pseudo-valid formulas.

Lemma 4.6 *Let $\mathfrak{M} = (W, \preceq, R_1, \ldots, R_{\ell+1}, V)$ be an EK-structure of order $(\ell+1)$. Then we have for all $(a,i) \in W^\sharp$ and all α that*

$$(\mathfrak{M}, a) \models^{ps} \alpha \quad \Longleftrightarrow \quad (\mathfrak{M}^\sharp, (a,i)) \models^{ps} \alpha.$$

Proof. We show this claim by induction on the structure of α and distinguish the following cases.

(i) α is the logical constant $\bot$ or an atomic proposition. Then the situation is clear.

(ii) α is a disjunction or a conjunction. Then we simply have to apply the induction hypothesis.

(iii) α is of the form $\beta \rightarrow \gamma$. To show the direction from left to right we assume $(\mathfrak{M}, a) \models^{ps} \beta \rightarrow \gamma$ and thus have

$$(\mathfrak{M}, b) \models^{ps} \beta \quad \Longrightarrow \quad (\mathfrak{M}, b) \models^{ps} \gamma \quad \text{for all } b \text{ such that } a \preceq b. \tag{1}$$

In order to prove $(\mathfrak{M}^\sharp, (a,i)) \models^{ps} \beta \rightarrow \gamma$, we pick an arbitrary $(c,j) \in W^\sharp$ for which $(a,i) \preceq^\sharp (c,j)$ and $(\mathfrak{M}^\sharp, (c,j)) \models^{ps} \beta$. By the induction hypothesis we obtain $(\mathfrak{M}, c) \models^{ps} \beta$, and in view of the definition of $\preceq^\sharp$ we also have $a \preceq c$. Hence (1) gives us $(\mathfrak{M}, c) \models^{ps} \gamma$, and a further application of the induction hypothesis yields $(\mathfrak{M}^\sharp, (c,j)) \models^{ps} \gamma$, as we had to show. The proof of the converse directions follows exactly the same pattern.

(iv) α is of the form $\mathsf{K}_j(\beta)$. For establishing the direction from left to right assume $(\mathfrak{M}, a) \models^{ps} \mathsf{K}_j(\beta)$, yielding that

$$(\mathfrak{M}, b) \models^{ps} \beta \quad \text{for all } b \in R_j[a]. \tag{2}$$

Now we pick an arbitrary element (c,k) of $R_j^\sharp[(a,i)]$. According to the definition of $R_j^\sharp$ this implies that $c \in R_j[a]$, and in view of (2), we thus obtain $(\mathfrak{M}, c) \models^{ps} \beta$. Now we can apply the induction hypothesis and have $(\mathfrak{M}^\sharp, (c,k)) \models^{ps} \beta$. Therefore, $(\mathfrak{M}^\sharp, (a,i)) \models^{ps} \mathsf{K}_j(\beta)$.

For the converse direction we proceed from $(\mathfrak{M}^\sharp, (a,i)) \models^{ps} \mathsf{K}_j(\beta)$, i.e. from

$$(\mathfrak{M}^\sharp, (b,j)) \models^{ps} \beta \quad \text{for all } (b,j) \in R_j^\sharp[(a,i)]. \tag{3}$$

Given any element c of $R_j[a]$, we obtain $(c,j) \in R_j^\sharp[(a,i)]$, thus that (3) implies $(\mathfrak{M}^\sharp, (c,j)) \models^{ps} \beta$. Applying the induction hypothesis then immediately leads to $(\mathfrak{M}, c) \models^{ps} \beta$. Hence we have $(\mathfrak{M}, a) \models^{ps} \mathsf{K}_j(\beta)$.

(v) α is of the form $\mathsf{D}(\beta)$. This case can be handled as the previous case since D is interpreted by the relations $R_{\ell+1}$ and $R^\sharp_{\ell+1}$, respectively. $\Box$

An immediate consequence of this lemma is that the property of being a (**D2**)-pseudo-model or a (**D2T**)-pseudo-model is inherited from an EK-structure $\mathfrak{M}$ to its strict extension $\mathfrak{M}^\sharp$.

Corollary 4.7 *If $\mathfrak{M}$ is a (**D2**)-pseudo-model, then $\mathfrak{M}^\sharp$ is a (**D2**)-pseudo-model as well; if $\mathfrak{M}$ is a (**D2T**)-pseudo-model, then also $\mathfrak{M}^\sharp$ is a (**D2T**)-pseudo-model.*

The strict extensions of (**D2**)-pseudo-models have a further property that will be needed in the proof of Lemma 4.10.

Lemma 4.8 *Let $\mathfrak{M} = (W, \preceq, R_1, \dots, R_{\ell+1}, V)$ be a (**D2**)-pseudo-model and j one of the numbers $1, \dots, \ell$. Then we have for all $(a, i), (b, k) \in W^\sharp$ and all α that*

$$\left.\begin{array}{l}(\mathfrak{M}^\sharp, (a, i)) \models^{ps} \mathsf{K}_j(\alpha) \;\; \textit{and} \\ (b, k) \in (R^\sharp_j \cup R^\sharp_{\ell+1})[(a, i)]\end{array}\right\} \quad \Longrightarrow \quad (\mathfrak{M}^\sharp, (b, k)) \models^{ps} \alpha.$$

Proof. Since $\mathfrak{M}^\sharp$ is a (**D2**)-pseudo-model, $(\mathfrak{M}^\sharp, (a, i)) \models^{ps} \mathsf{D}(\alpha)$ follows from the assumption $(\mathfrak{M}^\sharp, (a, i)) \models^{ps} \mathsf{K}_j(\alpha)$. In this pseudo-model the operator D is interpreted by means of the accessibility relation $R^\sharp_{\ell+1}$, hence the conclusion is an immediate consequence. $\Box$

EK-structures of order $(\ell+1)$ provide only intermediate tools for the canonical model construction. In the end we are interested in EK-stuctures of order ℓ, and in order to build those, we now collapse EK-structures $\mathfrak{M}$ of order $(\ell+1)$ to so-called associated structures $\mathfrak{M}^\star$ of order ℓ, via their strict extensions $\mathfrak{M}^\sharp$.

Definition 4.9 [Associated structure] Given an EK-structure

$$\mathfrak{M} = (W, \preceq, R_1, \dots, R_{\ell+1}, V)$$

of order $(\ell + 1)$, the structure *associated with* $\mathfrak{M}$ is defined to be the structure

$$\mathfrak{M}^\star \; = \; (W^\star, \preceq^\star, R^\star_1, \dots, R^\star_\ell, V^\star),$$

where we set:

($\star$1) $W^\star \; := \; W^\sharp, \quad \preceq^\star := \preceq^\sharp, \quad V^\star \; := \; V^\sharp$,

($\star$2) $R^\star_i \; := \; R^\sharp_i \cup R^\sharp_{\ell+1} \quad$ for $i = 1, \dots, \ell$.

It is clear that $\mathfrak{M}^\star$ is an EK-structure of order ℓ. The decisive property of this construction is that validity with respect to the structure associated with an EK-structure $\mathfrak{M}$ of order $(\ell + 1)$ coincides with pseudo-validity with respect to its strict extension $\mathfrak{M}^\sharp$.

Lemma 4.10 *Given a (**D2**)-pseudo-model $\mathfrak{M} = (W, \preceq, R_1, \dots, R_{\ell+1}, V)$, we have for all $(a, i) \in W^\sharp$ and all α that*

$$(\mathfrak{M}^\star, (a, i)) \models \alpha \quad \Longleftrightarrow \quad (\mathfrak{M}^\sharp, (a, i)) \models^{ps} \alpha.$$

Proof. The proof of this equivalence is by induction on the structure of α. We distinguish the following cases:

(i) α is the logical constant $\bot$ or an atomic proposition. Then the claim follows immediately.

(ii) α is a disjunction, a conjunction, or an implication. Then we simply have to apply the induction hypothesis.

(iii) α is of the form $\mathsf{K}_j(\beta)$. In view of the definition of $R_j^\star$, the direction from left to right is obtained by a straightforward application of the induction hypothesis. For proving the converse direction, assume $(\mathfrak{M}^\sharp, (a,i)) \models^{ps} \mathsf{K}_j(\beta)$. However, then Lemma 4.8 implies $(\mathfrak{M}^\sharp, (b,k)) \models^{ps} \beta$ for all elements (b,k) of $(R^\sharp \cup R^\sharp_{\ell+1})[(a,i)] = R_j^\star[(a,i)]$. For all those (b,k) the induction hypothesis yields $(\mathfrak{M}^\star, (b,k)) \models \beta$, and thus we have $(\mathfrak{M}^\star, (a,i)) \models \mathsf{K}_j(\beta)$.

(iv) α is of the form $\mathsf{D}(\beta)$. Now we observe that

$$\begin{aligned}\textstyle\bigcap_{j=1}^{\ell} R_j^\star[(a,i)] &= \textstyle\bigcap_{j=1}^{\ell}(R_j^\sharp \cup R^\sharp_{\ell+1})[(a,i)]\\ &= (\textstyle\bigcap_{j=1}^{\ell} R_j^\sharp[(a,i)]) \cup R^\sharp_{\ell+1}[(a,i)] \;=\; R^\sharp_{\ell+1}[(a,i)],\end{aligned}$$

where the last equality follows from Lemma 4.5. Hence the operator D is interpreted in $\mathfrak{M}^\star$ as in $\mathfrak{M}^\sharp$, and our assertion is immediate from the induction hypothesis. ☐

We now come to the main theorem of this section. It is an immediate consequence of Lemma 4.6 and the previous lemma.

Theorem 4.11 *If $\mathfrak{M} = (W, \preceq, R_1, \ldots, R_{\ell+1}, V)$ is a **(D2)**-pseudo-model, then we have for all $(a,i) \in W^\sharp$ and all α that*

$$(\mathfrak{M}, a) \models^{ps} \alpha \quad \Longleftrightarrow \quad (\mathfrak{M}^\star, (a,i)) \models \alpha.$$

*In particular, if $\mathfrak{M}$ is a **(D2T)**-pseudo-model, then $\mathfrak{M}^\star$ is a **(T)**-model.*

5 Prime sets and completeness

Now we introduce syntactic EK-structures that are based on so-called prime sets. This is a standard approach to proving completeness of intuitionistic modal systems also used in, for example, Fischer Servi [12] and Simpson [13].

Recall that **ID•** stands for one of the theories **IDK** or **IDT**. If P is a set of formulas, then we write $P \vdash_{\mathbf{ID\bullet}} \beta$ iff there exist finitely many formulas $\gamma_1, \ldots, \gamma_n \in P$ such that $\mathbf{ID\bullet} \vdash (\gamma_1 \wedge \ldots \wedge \gamma_n) \to \beta$.

Definition 5.1 [Prime set] A set P of formulas is called **ID•**-*prime* iff it satisfies the following conditions:

(P.1) $P \vdash_{\mathbf{ID\bullet}} \beta \quad \Longrightarrow \quad \beta \in P$,

(P.2) $\beta \vee \gamma \in P \quad \Longrightarrow \quad \beta \in P$ or $\gamma \in P$,

(P.3) $\bot \notin P$.

The following prime lemma describes a crucial property of prime sets. Its proof is standard and similar to that in Jäger and Marti [7].

Lemma 5.2 (Prime lemma) *Suppose that $P \nvdash_{\mathbf{ID}\bullet} \alpha$ for some set of formulas P and some α. Then there exists an* **ID•***-prime set Q such that $P \subseteq Q$ and $Q \nvdash_{\mathbf{ID}\bullet} \alpha$.*

Relative to the theory **ID•** we now introduce the canonical EK-structure $\mathfrak{C}$. In order to keep the notation readable, we refrain from explicitly mentioning **ID•** (for example as sub- or superscript), but it should always be clear from the context to which theory we refer. If N is a set of formulas and H one of the modal operators $\mathsf{K}_1, \ldots, \mathsf{K}_\ell$, or D, we write $\mathsf{H}^{-1}(N)$ for $\{\gamma : \mathsf{H}(\gamma) \in N\}$.

Definition 5.3 [Canonical structure] The *canonical structure for* **ID•** is the $(\ell + 4)$ tuple

$$\mathfrak{C} \; = \; (\mathcal{W}, \subseteq, \mathcal{R}_1, \ldots, \mathcal{R}_{\ell+1}, \mathcal{V}),$$

where we define:

(Can1) $\mathcal{W} \; := \; \{P : P \text{ is an } \mathbf{ID}\bullet\text{-prime set of formulas}\}$,

(Can2) For any $i = 1, \ldots, \ell$: $\mathcal{R}_i \; := \; \{(P, Q) \in \mathcal{W} \times \mathcal{W} : \mathsf{K}_i^{-1}(P) \subseteq Q\}$,

(Can3) $\mathcal{R}_{\ell+1} \; := \; \{(P, Q) \in \mathcal{W} \times \mathcal{W} : \mathsf{D}^{-1}(P) \subseteq Q\}$,

(Can4) $\mathcal{V}$ is the function from $\mathcal{W}$ to the power set of $PROP$ given by

$$\mathcal{V}(P) \; := \; \{p : p \in P\}.$$

It is evident that $\mathfrak{C}$ is an EK-structure of order $(\ell + 1)$. All further relevant properties follow more or less directly from the following truth property.

Lemma 5.4 (Truth lemma) *Let $\mathfrak{C} = (\mathcal{W}, \subseteq, \mathcal{R}_1, \ldots, \mathcal{R}_{\ell+1}, \mathcal{V})$ be the canonical structure for* **ID•***. Then we have for all α and all $P \in \mathcal{W}$ that*

$$\alpha \in P \quad \Longleftrightarrow \quad (\mathfrak{C}, P) \models^{ps} \alpha.$$

Proof. We establish this equivalence by induction on the structure of α and distinguish the following cases.

(i) It trivially holds in case that α is the logical constant $\bot$ or an atomic proposition.

(ii) If α is a disjunction or a conjunction it follows from the induction hypothesis and the properties of **ID•**-prime sets.

(iii) α is of the form $\beta_1 \rightarrow \beta_2$. We first assume that

$$\beta_1 \rightarrow \beta_2 \in P, \qquad P \subseteq Q \in \mathcal{W}, \qquad \text{and} \qquad (\mathfrak{C}, Q) \models^{ps} \beta_1.$$

Then we have $\beta_1 \rightarrow \beta_2 \in Q$ and (by the induction hypothesis) $\beta_1 \in Q$. Since Q is deductively closed, this yields $\beta_2 \in Q$ and thus again by the induction hypothesis that $(\mathfrak{C}, Q) \models^{ps} \beta_2$. Q has been an arbitrary superset of P within $\mathcal{W}$, and thus we conclude $(\mathfrak{C}, P) \models^{ps} \beta_1 \rightarrow \beta_2$.

Now assume $(\mathfrak{C}, P) \models^{ps} \beta_1 \to \beta_2$ and $\beta_1 \to \beta_2 \notin P$. Since P is deductively closed, we have $P \cup \{\beta_1\} \nvdash_{\mathbf{ID\bullet}} \beta_2$. By the prime lemma there exists a $Q \in \mathcal{W}$ such that

$$P \cup \{\beta_1\} \subseteq Q \quad \text{and} \quad Q \nvdash_{\mathbf{ID\bullet}} \beta_2, \quad \text{hence} \quad \beta_2 \notin Q.$$

Together with the induction hypothesis we thus obtain

$$(\mathfrak{C}, Q) \models^{ps} \beta_1 \quad \text{and} \quad (\mathfrak{C}, Q) \not\models^{ps} \beta_2.$$

Since $P \subseteq Q$, this contradicts $(\mathfrak{C}, P) \models^{ps} \beta_1 \to \beta_2$.

(iv) α is of the form $\mathsf{K}_i(\beta)$. For the direction from left to right assume

$$\mathsf{K}_i(\beta) \in P \quad \text{and} \quad \mathsf{K}_i^{-1}(P) \subseteq Q$$

for an arbitrary $Q \in \mathcal{W}$. This implies $\beta \in Q$, and in view of the induction hypothesis we thus have $(\mathfrak{C}, Q) \models^{ps} \beta$. Therefore, $(\mathfrak{C}, P) \models^{ps} \mathsf{K}_i(\beta)$.

For the converse direction we assume $(\mathfrak{C}, P) \models^{ps} \mathsf{K}_i(\beta)$. We first claim that

$$\mathsf{K}_i^{-1}(P) \vdash_{\mathbf{ID\bullet}} \beta. \tag{*}$$

To establish this claim, assume for contradiction that $\mathsf{K}_i^{-1}(P) \nvdash_{\mathbf{ID\bullet}} \beta$. According to the prime lemma we thus have a $Q \in \mathcal{W}$ such that $\mathsf{K}_i^{-1}(P) \subseteq Q$ and $Q \nvdash_{\mathbf{ID\bullet}} \beta$. In particular, $\beta \notin Q$. By the induction hypothesis, this yields $(\mathfrak{C}, Q) \not\models^{ps} \beta$; a contradiction to $(\mathfrak{C}, P) \models^{ps} \mathsf{K}_i(\beta)$ and $\mathsf{K}_i^{-1}(P) \subseteq Q$.

From (*) we conclude that there are $\gamma_1, \dots, \gamma_n \in \mathsf{K}_i^{-1}(P)$ such that

$$\mathbf{ID\bullet} \vdash (\gamma_1 \wedge \ldots \wedge \gamma_n) \to \beta.$$

Thus we also have

$$\mathbf{ID\bullet} \vdash (\mathsf{K}_i(\gamma_1) \wedge \ldots \wedge \mathsf{K}_i(\gamma_n)) \to \mathsf{K}_i(\beta),$$

with $\mathsf{K}_i(\gamma_1), \dots, \mathsf{K}_i(\gamma_n) \in P$, implying that $P \vdash_{\mathbf{ID\bullet}} \mathsf{K}_i(\beta)$. Hence $\mathsf{K}_i(\beta) \in P$ since P is deductively closed.

(v) α is of the form $\mathsf{D}(\beta)$. Because of the pseudo-validity interpretation of D, this case is treated exactly as the previous cases. □

Corollary 5.5

(i) *If $\mathfrak{C}$ is the canonical structure for* **IDK***, then $\mathfrak{C}$ is a (**D2**)-pseudo-model.*

(ii) *If $\mathfrak{C}$ is the canonical structure for* **IDT***, then $\mathfrak{C}$ is a (**D2T**)-pseudo-model.*

Proof. We only have to remember that an **IDK**-prime set of formulas P is deductively closed with respect to derivability in **IDK** and, therefore, contains $\mathsf{K}_i(\alpha) \to \mathsf{D}(\alpha)$ for all $i = 1, \dots, \ell$ and all α. Analogously, any **IDT**-prime set of formulas Q contains, in addition, the formulas $\mathsf{D}(\alpha) \to \alpha$ for any α. Thus the truth lemma implies our assertions. □

Now the stage is set, and combining what we have obtained so far, we can state the following first main result.

Theorem 5.6 *Let* $\mathfrak{C} = (\mathcal{W}, \subseteq, \mathcal{R}_1, \ldots, \mathcal{R}_{\ell+1}, \mathcal{V})$ *be the canonical structure for* **ID•** *and* $\mathfrak{C}^\star$ *the EK-structure of order* ℓ *associated with* $\mathfrak{C}$*. Then we have for all* **ID•***-prime sets of formulas* P*, all* α*, and all* $i = 1, \ldots, \ell$ *that*

$$\alpha \in P \quad \Longleftrightarrow \quad (\mathfrak{C}^\star, (P, i)) \models \alpha.$$

Proof. In view of the truth lemma and Lemma 4.6 we have

$$\alpha \in P \quad \Longleftrightarrow \quad (\mathfrak{C}, P) \models^{ps} \alpha \quad \Longleftrightarrow \quad (\mathfrak{C}^\sharp, (P, i)) \models^{ps} \alpha$$

for the strict extension $\mathfrak{C}^\sharp$ of $\mathfrak{C}$. Furthermore, $\mathfrak{C}^\sharp$ is a (**D2**)-pseudo-model according to the previous corollary. Hence we can apply Lemma 4.10 and see that

$$(\mathfrak{C}^\star, (P, i)) \models \alpha \quad \Longleftrightarrow \quad (\mathfrak{C}^\sharp, (P, i)) \models^{ps} \alpha.$$

Therefore, we have what we want. ◻

Theorem 5.7 (Completeness) *For all* α *we have:*

(i) $\models \alpha \quad \Longrightarrow \quad \mathbf{IDK} \vdash \alpha.$

(ii) $\models_{ref} \alpha \quad \Longrightarrow \quad \mathbf{IDT} \vdash \alpha.$

Proof. For the first assertion, assume $\models \alpha$ and $\mathbf{IDK} \nvdash \alpha$. Note that then the prime lemma thus tells us that there exists an **IDK**-prime set P for which $P \nvdash_{\mathbf{IDK}} \alpha$. Hence $\alpha \notin P$. Consider the canonical structure $\mathfrak{C}$ for **IDK** and the EK-structure $\mathfrak{C}^\star$ associated with $\mathfrak{C}$. According to Theorem 5.6 we have $(\mathfrak{C}^\star, (P, i)) \not\models \alpha$ for $i = 1, \ldots, \ell$. This is a contradiction to $\models \alpha$.

We come to the second assertion. Now we assume $\models_{ref} \alpha$ and $\mathbf{IDT} \nvdash \alpha$. In this case the prime lemma gives us an **IDT**-prime set Q for which $Q \nvdash_{\mathbf{IDK}} \alpha$ and, consequently, $\alpha \notin Q$. Now we work with the canonical structure $\mathfrak{C}$ for **IDT** and the EK-structure $\mathfrak{C}^\star$ associated with $\mathfrak{C}$. We see that $\mathfrak{C}$ is a (**D2T**)-pseudomodel by Corollary 5.5 and, consequently, $\mathfrak{C}^\star$ is a (**T**)-model by Theorem 4.11. In view of Theorem 5.6 we also have $(\mathfrak{C}^\star, (Q, i)) \not\models \alpha$ for any $i = 1, \ldots, \ell$. It only remains to move to the reflexive extension $\overline{\mathfrak{C}^\star}$ of $\mathfrak{C}^\star$ and to apply Lemma 3.3. It follows that $(\overline{\mathfrak{C}^\star}, (Q, i)) \not\models \alpha$. Since $\overline{\mathfrak{C}^\star}$ is reflexive, this is a contradiction to $\models_{ref} \alpha$. ◻

Together with Theorem 3.1 we thus have that **IDK** and **IDT** are sound and complete formalizations of intuitionistic distributed knowledge. In work in progress extensions of these results to systems including positive introspection and common knowledge as well as the question of the finite model property will be considered.

As in classical epistemic logic, positive introspection can be formalized by the axioms $\mathsf{D}(\alpha) \to \mathsf{D}(\mathsf{D}(\alpha))$ and $\mathsf{K}_i(\alpha) \to \mathsf{K}_i(\mathsf{K}_i(\alpha))$, where the latter semantically corresponds to the transitivity of the relations R_i. We think that an approach to completeness of intuitionistic **S4** with distributed knowledge can be done along the lines of the constructions in Fagin, Halpern, and Vardi [3] and Wang and Ågotnes [15]. Negative introspection is less straightforward, as intuitionistic **S5** is typically formulated by making use of the box and the diamond operator; see, e.g., Fischer Servi [12] and Simpson [13], and in intuitionistic modal logic $\Diamond(\alpha)$ is not equivalent to $\neg\Box(\neg\alpha)$. In our present framework the operators $\mathsf{K}_1, \ldots, \mathsf{K}_\ell$ correspond to boxes. But the corresponding diamonds are not available.

References

[1] Artemov, S. and T. Protopopescu, *Intuitionistic Epistemic Logic*, arXiv:1406.1582v4 [math.LO] (2014).

[2] Fagin, R., J. Halpern, Y. Moses and M. Vardi, "Reasoning about Knowledge," MIT Press, 1995.

[3] Fagin, R., J. Halpern and M. Vardi, *What can machines know? On the properties of knowledge in distributed systems*, Journal of the Association for Computing Machinery **39** (1992), pp. 328–376.

[4] Gerbrandy, J., *Distributed knowledge*, in: J. Hulstijn and A. Nijholt, editors, *Formal Semantics and Pragmatics of Dialogue, Twendia '98* (1998), pp. 111–123.

[5] Hakli, R. and S. Negri, *Proof theory for distributed knowledge*, in: F. Sadri and K. Satoh, editors, *Computational Logic in Multi-Agent Systems*, Lecture Notes in Artificial Intelligence **5056** (2008), pp. 100–116.

[6] Hirai, Y., *An intuitionistic epistemic logic for sequential consistency on shared memory*, in: E. Clarke and A. Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning*, Lecture Notes in Computer Science **6355** (2010), pp. 272–289.

[7] Jäger, G. and M. Marti, *Intuitionistic common knowledge or belief*, to appear in Journal of Applied Logic.

[8] Krupski, V. and A. Yatmanov, *Sequent Calculus for Intuitionistic Epistemic Logic*, arXiv:1508.07851v1 [cs.LO] (2015).

[9] Meyer, J.-J. C. and W. van der Hoek, "Epistemic Logic for AI and Computer Science," Cambridge Tracts in Theoretical Computer Science **41**, Cambridge University Press, 2004.

[10] Plotkin, G. and C. Sterling, *A framework for intuitionistic modal logic,*, Theoretical Aspects of Reasoning About Knowledge (J.Y.Halpern, ed.), Morgan Kaufmann Publishers (1986), pp. 399–406.

[11] Proietti, C., *Intuitionistic epistemic logic, Kripke models and Fitch's paradox*, Journal of Philosophical Logic **41** (2012), pp. 877–900.

[12] Servi, G. F., *Axiomatizations for some intuitionistic modal logics*, Rendiconti del Seminario Matematico Università Politecnico di Torino **42** (1984), pp. 179–194.

[13] Simpson, A., "The Proof Theory and Semantics of Intuitionistic Modal Logic (revised version)," Ph.D. thesis, University of Edinburgh (1994).

[14] Suzuki, N., *Semantics for intuitionistic epistemic logics of shallow depths for game theory*, Economic Theory **53** (2012), pp. 85–110.

[15] Wáng, Y. and T. Ågotnes, *Public announcement logic with distributed knowledge*, in: H. van Ditmarsch, J. Lang and S. Ju, editors, *LORI 2011*, Lecture Notes in Artificial Intelligence **6953** (2011), pp. 328–341.

[16] Williamson, T., *On intuitionistic modal epistemic logic*, Journal of Philosophy **21** (1992), pp. 63–89.