Classical and Empirical Negation in Subintuitionistic Logic

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Abstract

Subintuitionistic (propositional) logics are those in a standard intuitionistic language that result by weakening the frame conditions of the Kripke semantics for intuitionistic logic. In this paper we consider two negation expansions of subintuitionistic logic, one by classical negation and the other by what has been dubbed "empirical" negation. We provide an axiomatization of each expansion and show them sound and strongly complete. We conclude with some final remarks, including avenues for future research.

Keywords: Subintuitionistic logic, intuitionistic logic, classical negation, empirical negation

1 Introduction

With the advent of Kripke semantics for modal logic, it became natural to investigate logics weaker than **S5** by weakening its frame properties. For intuitionistic logic things have tended in the other direction through the investigation of intermediate logics that obtain by strengthening the frame properties of intuitionistic logic. There has been less interest in subintuitionistic logics that obtain by weakening the frame properties of intuitionistic logic. To our knowledge, there has been no study of expansions of subintuitionistic logic by operators outside of the standard language. Our interest lies in expansions of subintuitionistic logic that obtain by adding classical or classical-like negations

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to the language. Since we are interested in logics as consequence relations, our work most closely resembles Greg Restall's [14]. [3]

In Kripke semantics, classical negation is ordinarily defined such that the negation $\neg A$ of a sentence is true at a state just in case A is not true at that state. In this sense classical negation is extensional, since it does not require looking beyond the state of evaluation. The problem with this definition, however, is that it is incompatible with Heredity, i.e., that if an atomic statement A is true at some state w, then it is true at any state accessible to w. For if A is not true at x so that $\neg A$ is, and if y is accessible from x, we cannot require that the truth of $\neg A$ be preserved up to y since A may be true there. So adding classical negation requires either eschewing Heredity, or else restricting the accessibility relation in a way that allows Heredity to stay in force. The latter option is not available in the case of intuitionistic logic since there are no restrictions that would preserve the intuitionistic fragment. For instance, one natural restriction is to let accessibility be identity but that restriction results in classical logic. [4] We must therefore eschew Heredity, which makes subintuitionistic logic ideal for the purposes of adding classical negation.

There is another way of adding a classical-like negation to intuitionistic logic. A defining characteristic of classicality is that $\neg A$ is true just in case A is not. We may then define truth in a model to be truth at a distinguished base state, and define negation so that $\neg A$ is true in a model if, and only if, A is not. This means that negation is no longer extensional in the sense that the truth of $\neg A$ at an aribtrary *state* in a model depends on whether A is true at the base state, yet it is extensional in the sense that determining whether a negation is true in a model requires going nowhere else besides the state relative to which truth-in-a-model is defined, namely, the base state. Notice that, importantly, this definition of negation is compatible with Heredity. Suppose g is the base state of a model and that A is not true there. Then $\neg A$ is true at *every* state in the model, and so its truth is trivially preserved up the accessibility relation.

In [3] and [4], Michael De and Hitoshi Omori give philosophical grounds for adding such a negation to the language of intuitionistic logic, where they there call it *empirical negation*. To put matters briefly, intuitionistic negation, standardly defined as implication to absurdity, is too strong to allow for a generalization of intuitionistic logic to empirical domains, such as the physical sciences. [5] For instance, to express that a certain proposition (such as Goldbach's conjecture) has *not* been proven, it is too strong to say that the supposition of the proposition leads to absurdity. The idea, then, is to expand the vocabulary with an empirical negation that roughly expresses that "There fails to be sufficient evidence at present to warrant the proposition that...".

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[3] See [2], [6], [1] and the references cited therein for work on subintuitionistic logic.

[4] Compare this with classical relevance logic which results by adding classical negation to relevance logic in just this way, i.e. by letting the order relative to which truth is preserved be identity. Here, however, we do not get a collapse to classical logic. See [10] and [11] for details.

[5] This project is given its fullest defense in the works of Michael Dummett.

The negation has a number of interesting properties. One in particular is its close affinity to classical negation even though adding it to intuitionistic logic does not result in a collapse to classical logic.[6]

The paper proceeds as follows. In §2 we present the weakest subintuitionistic logic **SJ** of [14]. In §3 we expand **SJ** by empirical negation and show the axiomatization sound and strongly complete. In §4 we do the same for classical negation. In §5 we draw some comparisons to related systems, including weak relevance logics and a system of [5] in which both classical and intuitionistic negation coexist. In §6 we conclude with some final remarks.

2 Subintuitionistic logics revisited

We begin by presenting the weakest of the subintuitionistic logics, **SJ** which we then go onto expand by empirical and classical negation respectively in §3 and §4.

Definition 2.1 The language $\mathcal{L}$ consists of a finite set $\{\wedge, \vee, \rightarrow\}$ of propositional connectives and a denumerable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form the set of formulae defined as usual in $\mathcal{L}$. We denote a formula of $\mathcal{L}$ by A, B, C, etc. and a set of formulae of $\mathcal{L}$ by Γ, Δ, Σ, etc.

2.1 Semantics

Definition 2.2 A model for the language $\mathcal{L}$ is a quadruple $\langle W, g, R, V\rangle$, where W is a non-empty set (of states); $g \in W$ (the base state); R is a binary relation on W satisfying what in [14] is called *omniscience*, namely gRw for all $w \in W$; and $V : W \times \mathsf{Prop} \rightarrow \{0, 1\}$ an assignment of truth values to state-variable pairs.[7] Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$
- $I(w, A \rightarrow B) = 1$ iff for all $x \in W$, if wRx and $I(x, A) = 1$ then $I(x, B) = 1$.

Semantic consequence is now defined in terms of truth preservation at g: $\Sigma \models A$ iff for all models $\langle W, g, R, I\rangle$, $I(g, A) = 1$ if $I(g, B) = 1$ for all $B \in \Sigma$.

Remark 2.3 Note that we are not assuming

(Heredity) If $V(w, p) = 1$ and wRx then $V(x, p) = 1$,

[6] However, see [7] and [5] for different approaches to combining both classical and intuitionistic negation without such a collapse. It should be noted that the natural deduction system presented in [7] is trivial as stated (i.e. any formula follows from any set of formulae), an error that is to be blamed on *EFQ*¬. To see this let Γ be empty and α a contradiction. The correct rule is obtained by deleting α from the left-hand side of the premise sequent (as Lloyd Humberstone conveyed in personal communication).

[7] Omniscient frames are sometimes referred to as *strongly generated*.

nor that R is reflexive or transitive; R is assumed only to satisfy omniscience. Typically, inuitionistic Kripke models are not pointed in the sense that they contain a base state relative to which truth in the model is defined. It plays an essential role here for logics—taken as consequence relations and not classes of theorems—properly weaker than intuitionistic logic, but also for intuitionistic logic expanded by empirical negation.

2.2 Proof Theory

Definition 2.4 The system **SJ** consists of the following axiom schemata.

(Ax1) $A \to A$

(Ax2) $A \to (B \to B)$

(Ax3) $((A \to B) \wedge (B \to C)) \to (A \to C)$

(Ax4) $(A \wedge B) \to A$

(Ax5) $(A \wedge B) \to B$

(Ax6) $((C \to A) \wedge (C \to B)) \to (C \to (A \wedge B)))$

(Ax7) $A \to (A \vee B)$

(Ax8) $B \to (A \vee B)$

(Ax9) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C))$

(Ax10) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

In addition to these axioms, we have the following rules of inference.

(MP) $\dfrac{A \quad A \to B}{B}$ (DMP) $\dfrac{A \vee C \quad (A \to B) \vee C}{B \vee C}$

(Adj) $\dfrac{A \quad B}{A \wedge B}$ (DR) $\dfrac{(A \to B) \vee E \quad (C \to D) \vee E}{((B \to C) \to (A \to D)) \vee E}$

Finally, we write $\Gamma \vdash A$ if there is a sequence of formulae $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of **SJ**; (iii) is obtained by one of the rules (MP)–(DR) from formulae preceding it in sequence.

Remark 2.5 Note that the following rule, included in the original formulation of **SJ**, is derivable in view of (DR), (MP), (Ax1) and (Ax9).

(R) $$\frac{A \to B \quad C \to D}{(B \to C) \to (A \to D)}$$

To see this, assume $A \to B$ and $C \to D$. Then by (Ax7) and (MP) we obtain $(A \to B) \vee E$ and $(C \to D) \vee E$ where E is $(B \to C) \to (A \to D)$. Then by applying (DR), we have $E \vee E$, and by (MP), (Ax1) and (Ax9), we obtain E, as desired.

Remark 2.6 It deserves noting that the following rules, known as Prefixing, Suffixing and Transitivity respectively, are derivable in **SJ** in view of (R), (MP) and (Ax1):

(Prefixing) $$\frac{C \to D}{(A \to C) \to (A \to D)}$$

(Suffixing) $$\frac{A \to B}{(B \to C) \to (A \to C)}$$

(Transitivity) $$\frac{A \to B \quad B \to C}{A \to C}$$

Proposition 2.7 *The following formulae and rules are provable in* **SJ**.

(1) $$\frac{A \vee B \quad A \to C}{C \vee B}$$

(2) $$((A \vee C) \wedge (B \vee C)) \to ((A \wedge B) \vee C)$$

Proof. Left as an exercise for the reader. □

2.3 Soundness and completeness

Theorem 2.8 (Soundness) *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \vdash A$ *then* $\Gamma \models A$.

Proof. The proof, by induction on the length of proof, can be found in [14]. The omniscience of g in used in showing that (DR) preserves validity. □

Following [14], the following notions will be used in the proofs of completeness.

Definition 2.9

(i) If Π is a set of sentences, let $\Pi_{\to}$ be the set of all members of Π of the form $A \to B$.
(ii) $\Sigma \vdash_{\Pi} A$ iff $\Sigma \cup \Pi_{\to} \vdash A$.
(iii) Σ is a Π-*theory* iff:
  (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$
  (b) if $\vdash_{\Pi} A \to B$ then (if $A \in \Sigma$ then $B \in \Sigma$).
(iv) Σ is *prime* iff (if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$).
(v) If X is any set of sets of formulae the binary relation R on X is defined thus:

$$\Sigma R \Delta \text{ iff (if } A \to B \in \Sigma \text{ then (if } A \in \Delta \text{ then } B \in \Delta)).$$

(vi) $\Sigma \vdash_{\Pi} \Delta$ iff for some $D_1, \dots, D_n \in \Delta, \Sigma \vdash_{\Pi} D_1 \vee \dots \vee D_n$.
(vii) $\vdash_{\Pi} \Sigma \to \Delta$ iff for some $C_1, \dots, C_n \in \Sigma$ and $D_1, \dots, D_m \in \Delta$:

$$\vdash_{\Pi} C_1 \wedge \dots \wedge C_n \to D_1 \vee \dots \vee D_m.$$

(viii) Σ is Π-*deductively closed* iff (if $\Sigma \vdash_{\Pi} A$ then $A \in \Sigma$).
(ix) $\langle \Sigma, \Delta \rangle$ is a Π-*partition* iff (i) $\Sigma \cup \Delta = \mathsf{Form}$ and (ii) $\nvdash_{\Pi} \Sigma \to \Delta$.

In all the above, if $\Pi = \emptyset$, then the prefix 'Π-' will simply be omitted.

With these notions in mind, some lemmas are listed without their proofs. For details, see [14].

Proposition 2.10 *If $A \vdash C$ and $B \vdash C$, then $A \vee B \vdash C$.*

Lemma 2.11 *If $\langle \Sigma, \Delta \rangle$ is a Π-partition then Σ is a prime Π-theory.*

Lemma 2.12 *If $\Sigma \not\vdash \Delta$ then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a partition, and Σ' is deductively closed.*

Corollary 2.13 *If $\Sigma \not\vdash A$ then there is $\Pi \supseteq \Sigma$ such that $A \notin \Pi$, Π is a prime Π-theory and Π is Π-deductively closed.*

Lemma 2.14 *If $\not\vdash_{\Pi} \Sigma \to \Delta$ then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a Π-partition.*

Lemma 2.15 *Let Σ be a prime Π-theory and $A \to B \notin \Sigma$. Then there is a prime Π-theory, Δ such that $\Sigma R \Delta$, $A \in \Delta$, $B \notin \Delta$.*

We are now in a position to prove completeness.

Theorem 2.16 (Completeness) *For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models A$ then $\Gamma \vdash A$.*

Proof. The proof, due to [14], is given in the appendix. □

Remark 2.17 Restall notes the following correspondences between frame conditions and valid formulae.

Frame conditions	Characteristic formulae
Heredity	$A \to (B \to A)$
Reflexivity	$(A \wedge (A \to B)) \to B$
Transitivity	$(A \to B) \to ((B \to C) \to (A \to C))$

We will make use of this result later.

Remark 2.18 As observed by Heinirich Wansing in [15], the sets of theorems for the systems of Corsi, Došen and Restall coincide. However, Corsi and Došen show only *weak* completeness using standard techniques from modal logics, whereas Restall shows *strong* completeness using techniques from relevance logics. Restall's axiomatization is better suited for our purposes since the other axiomatizations will not work when we expand the language by empirical negation (cf. Remark 3.6).

Remark 2.19 As we noted earlier, in a semantic setting there is no straightforward way of adding classical negation to intuitionistic logic since classical negation breaks Heredity, the preservation of truth up the accessibility relation. In subintuitionistic logics where Heredity is not in force, there is no obstacle to adding classical negation. Recall that Heredity is required for validating the schema $A \to (B \to A)$, and that this schema has been criticized for introducing fallacies of relevance, since the mere truth of A does not guarantee us that an arbitrary B relevantly implies A. This suggests that **SJ** gets us at least one step closer to relevance than full intuitionistic logic. What then is the relation between subintuitionistic and relevance logic? It is, to be exact, that **SJ** is the relevance logic $\mathbf{B}^{+}$ of [12] extended by axioms (Ax2) and (Ax3). [8]

[8] We note that (Ax3) characterizes one of the many further conditions on the ternary relation. See [13, Theorem 2] for the details.

Let us now turn to the Heredity-friendly way of adding a classical-like negation to subintutionistic logic, even though we are working with the weakest subintuitionistic logic without the Heredity axiom, $A \to (B \to A)$.

3 Empirical negation

Before turning to the proof theory of subintuitionistic logic with empirical negation, we start with some preliminaries.

Definition 3.1 The language $\mathcal{L}_\sim$ consists of a finite set $\{\sim, \wedge, \vee, \to\}$ of propositional connectives and a denumerable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by $\mathsf{Form}_\sim$ the set of formulae defined as usual in $\mathcal{L}_\sim$. We denote a formula of $\mathcal{L}_\sim$ by A, B, C, etc. and a set of formulae of $\mathcal{L}_\sim$ by Γ, Δ, Σ, etc.

3.1 Semantics

Definition 3.2 A model for the language $\mathcal{L}_\sim$ is a quadruple $\langle W, g, R, V\rangle$, where W is a non-empty set (of states); $g \in W$ (the base state); R is a binary relation on W with g omniscient; and $V : W \times \mathsf{Prop} \to \{0, 1\}$ an assignment of truth values to state-variable pairs. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$
- $I(w, \sim A) = 1$ iff $I(g, A) = 0$
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$
- $I(w, A \to B) = 1$ iff for all $x \in W$, if wRx and $I(x, A) = 1$ then $I(x, B) = 1$.

Semantic consequence is again defined in terms of truth preservation at g: $\Sigma \models_e A$ iff for all models $\langle W, g, R, I\rangle$, $I(g, A) = 1$ if $I(g, B) = 1$ for all $B \in \Sigma$.

Remark 3.3 Note that we are neither assuming Heredity, nor that R is reflexive or transitive; R is assumed only to satisfy omniscience.

3.2 Proof Theory

Definition 3.4 The system $\mathbf{SJ}^\sim$ is obtained by adding the following axiom schemata and rule of inference to $\mathbf{SJ}$:

(Ax$_\sim$1)	$A \vee \sim A$	(Ax$_\sim$4)	$\sim\sim\sim A \to \sim A$
(Ax$_\sim$2)	$B \to (\sim A \vee \sim\sim A)$	(Ax$_\sim$5)	$(\sim A \wedge \sim B) \to \sim(A \vee B)$
(Ax$_\sim$3)	$\sim\sim A \to (\sim A \to B)$	(DRP)	$\dfrac{(A \vee B) \vee C}{(\sim A \to \sim\sim B) \vee C}$

Finally, we write $\Gamma \vdash_e A$ if there is a sequence of formulae $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of $\mathbf{SJ}^\sim$; (iii) is obtained by one of the rules (MP)–(DRP) from formulae preceding it in sequence.

Remark 3.5 Note that the following rule is derivable in $\mathbf{SJ}^{\sim}$:

(RP) $$\frac{A \vee B}{{\sim}A \to {\sim}{\sim}B}$$

To see this, assume $A \vee B$. Then by (Ax7) and (MP), we obtain $(A \vee B) \vee ({\sim}A \to {\sim}{\sim}B)$, and by applying (DRP) we have $({\sim}A \to {\sim}{\sim}B) \vee ({\sim}A \to {\sim}{\sim}B)$ which implies ${\sim}A \to {\sim}{\sim}B$, as desired.

Remark 3.6 Note that even though (Ax$_{\sim}$1) is valid and that $B \models_e A \vee {\sim}A$, $B \to (A \vee {\sim}A)$ is *not* valid in $\mathbf{SJ}^{\sim}$. This is the reason both (Ax$_{\sim}$1) and (Ax$_{\sim}$2) are needed. This also shows that the rule of inference from A to $B \to A$, employed in the axiomatizations of Corsi and Došen, is *not* sound in $\mathbf{SJ}^{\sim}$.

3.3 Some Basic Results

Theorem 3.7 (Classical deduction theorem) *For* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}_{\sim}$*, if* $\Gamma, A \vdash_e B$ *then* $\Gamma \vdash_e {\sim}A \vee B$*.*

Proof. By induction on the length n of a proof of $\Gamma, A \vdash_e B$. The details are given in the appendix. □

For the purpose of proving the other direction of the deduction theorem, we need another lemma.

Lemma 3.8 *The following are derivable in* $\mathbf{SJ}^{\sim}$*:*

(RC$\sim$) $$\frac{A \to B}{{\sim}B \to {\sim}A}$$

(RIDN) $$\frac{A}{{\sim}{\sim}A}$$

(3) $${\sim}{\sim}(A \to A)$$

($\sim$-DS) $$\frac{A \quad {\sim}A \vee B}{B}$$

Proof. For (RC$\sim$), assume $A \to B$. Then by making use of (1) and (Ax$_{\sim}$1), we have $B \vee {\sim}A$. Thus by applying (RP), we obtain ${\sim}B \to {\sim}{\sim}{\sim}A$, and finally by (Ax$_{\sim}$4) and (Transitivity), we obtain ${\sim}B \to {\sim}A$, as desired.

For (3), first apply (RC$\sim$) to ${\sim}(A \to A) \to (A \to A)$, an instance of (Ax2), and we get ${\sim}(A \to A) \to {\sim}{\sim}(A \to A)$. This together with (Ax1) and (Adj) gives us $({\sim}(A \to A) \to {\sim}{\sim}(A \to A)) \wedge ({\sim}{\sim}(A \to A) \to {\sim}{\sim}(A \to A))$, and by (Ax9) and (MP), we obtain $({\sim}(A \to A) \vee {\sim}{\sim}(A \to A)) \to {\sim}{\sim}(A \to A)$. Thus, we obtain the desired result in view of (Ax$_{\sim}$1) and (MP).

For (RIDN), assume A. Then by (Ax8), we obtain ${\sim}(A \to A) \vee A$. By applying (RP), we get ${\sim}{\sim}(A \to A) \to {\sim}{\sim}A$. The desired result follows by this, (3) and (MP).

Finally, for ($\sim$-DS), assume A and ${\sim}A \vee B$. By the former and (RIDN), we obtain ${\sim}{\sim}A$, and by this together with (Ax$_{\sim}$3) and (MP), we obtain ${\sim}A \to B$. Therefore, by applying (1) to this and the latter assumption, we obtain $B \vee B$ which implies B, as desired. □

Remark 3.9 Note that ($\sim$-DS) implies *ex contradictione quodlibet*:

($\sim$-ECQ) $$\frac{A \quad {\sim}A}{B}$$

Proposition 3.10 *For* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}_{\sim}$*, if* $\Gamma \vdash_e {\sim}A \vee B$ *then* $\Gamma, A \vdash_e B$.

Proof. Immediate in view of (∼-DS). □

By combining Theorem 3.7 and Proposition 3.10, we obtain the following.

Theorem 3.11 *For* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}_{\sim}$*,* $\Gamma, A \vdash_e B$ *iff* $\Gamma \vdash_e {\sim}A \vee B$.

Corollary 3.12 *For* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}_{\sim}$*,* $\Gamma, A \vdash_e B$ *iff* $\Gamma \vdash_e {\sim}B \to {\sim}A$.

We note that the following are theorems of $\mathbf{SJ}^{\sim}$.

Proposition 3.13 *The following formulae are provable in* $\mathbf{SJ}^{\sim}$.

(4)	${\sim}(A \vee B) \to ({\sim}A \wedge {\sim}B)$	(6)	${\sim}A \to {\sim}{\sim}{\sim}A$
(5)	$({\sim}A \vee {\sim}B) \to {\sim}(A \wedge B)$	(7)	${\sim}A \to ({\sim}{\sim}A \to B)$

Proof. (4) and (5) are essentially proved using (Ax4), (Ax5) and (Ax7), (Ax8) respectively together with (RC∼). (6) follows immediately by (Ax$_{\sim}$1) and (RP). For (7), apply (Transitivity) to (6) and ${\sim}{\sim}{\sim}A \to ({\sim}{\sim}A \to B)$, an instance of (Ax$_{\sim}$3). □

3.4 Soundness and completeness

We now proceed to the proof of soundness and completeness.

Theorem 3.14 *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}_{\sim}$*, if* $\Gamma \vdash_e A$ *then* $\Gamma \models_e A$.

Proof. By induction on the length of the proof, as usual. □

Proposition 3.15 *If* $A \vdash_e C$ *and* $B \vdash_e C$*, then* $A \vee B \vdash_e C$.

Proof. Assume $A \vdash_e C$ and $B \vdash_e C$. Then, by Corollary 3.12, we obtain $\vdash_e {\sim}C \to {\sim}A$ and $\vdash_e {\sim}C \to {\sim}B$ respectively. By (Ax6), we get $\vdash_e {\sim}C \to ({\sim}A \wedge {\sim}B)$, and therefore $\vdash_e {\sim}C \to {\sim}(A \vee B)$ by (Ax$_{\sim}$5) and (MP). Finally, we obtain the desired result by another application of Corollary 3.12. □

The following lemmas are useful for the completeness proof.

Lemma 3.16 *If* Σ *is prime,* Π*-deductively closed and* $A \notin \Sigma$ *then* ${\sim}A \in \Sigma$.

Proof. If Σ is Π-deductively closed, then by (Ax$_{\sim}$1) we obtain $A \vee {\sim}A \in \Sigma$. This together with $A \notin \Sigma$ and the primeness of Σ implies ${\sim}A \in \Sigma$, as desired.□

Lemma 3.17 *If* Σ *is non-trivial, prime,* Π*-deductively closed and* $A \in \Sigma$ *then* ${\sim}{\sim}A \in \Sigma$.

Proof. Assume the required assumptions and suppose for reductio that ${\sim}A \in \Sigma$. Then this implies $\Sigma \vdash_\Pi {\sim}A$. Moreover, $A \in \Sigma$ implies that $\Sigma \vdash_\Pi A$. Therefore, these together with (∼-ECQ) imply that $\Sigma \vdash_\Pi B$ for any B, and since Σ is Π-deductively closed, we obtain $B \in \Sigma$ for any B. But this contradicts the assumption that B is non-trivial. Thus the given assumptions imply ${\sim}A \notin \Sigma$, and in view of Lemma 3.16, this implies ${\sim}{\sim}A \in \Sigma$, as desired. □

Lemma 3.18 *If* Σ *is a non-empty prime* Π*-theory and* ${\sim}A \notin \Sigma$ *then* ${\sim}{\sim}A \in \Sigma$.

Proof. Since Σ is non-empty, let B be an element of Σ. By (Ax$_{\sim}$2) we have $\vdash_e B \to (\sim A \vee \sim\sim A)$ and since Σ is a Π-theory, we obtain $\sim A \vee \sim\sim A \in \Sigma$. This together with $\sim A \notin \Sigma$ and the primeness of Σ imply $\sim\sim A \in \Sigma$, as desired. □

Theorem 3.19 *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}_{\sim}$*, if* $\Gamma \models_e A$ *then* $\Gamma \vdash_e A$.

Proof. The proof is given in the appendix. □

4 Classical negation

In [9], Ryo Kashima investigates some subintuitionistic logics with classical negation, but it is important to note two points of departure from the present work. The first is that Kashima proves only weak completeness, a point that is important in the present context and the reason that our (and Restall's) frames are omniscient. Doing away with omniscience yields a class of theorems equivalent to **SJ**'s, but a different consequence relation, hence a different logic in our sense. Second, Kashima works with sequent calculi, whereas we prefer to work with Hilbert-style axioms system.

Definition 4.1 The language $\mathcal{L}_{\neg}$ consists of a finite set $\{\neg, \wedge, \vee, \to\}$ of propositional connectives and a denumerable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by $\mathsf{Form}_{\neg}$ the set of formulae defined as usual in $\mathcal{L}_{\neg}$. We denote a formula of $\mathcal{L}_{\neg}$ by A, B, C, etc. and a set of formulae of $\mathcal{L}_{\neg}$ by Γ, Δ, Σ, etc.

4.1 Semantics

Definition 4.2 A model for the language $\mathcal{L}_{\neg}$ is a quadruple $\langle W, g, R, V\rangle$, where W is a non-empty set (of states); $g \in W$ (the base state); R is a binary relation on W with g omniscient; and $V : W \times \mathsf{Prop} \to \{0, 1\}$ an assignment of truth values to state-variable pairs. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$
- $I(w, \neg A) = 1$ iff $I(w, A) = 0$
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$
- $I(w, A \to B) = 1$ iff for all $x \in W$: if wRx and $I(x, A) = 1$ then $I(x, B) = 1$.

Semantic consequence is defined in terms of truth preservation at g: $\Sigma \models_c A$ iff for all models $\langle W, g, R, I\rangle$, $I(g, A) = 1$ if $I(g, B) = 1$ for all $B \in \Sigma$.

4.2 Proof Theory

Definition 4.3 The system $\mathbf{SJ}^{\neg}$ is obtained by adding the following axiom schemata to **SJ**.

(Ax$_{\neg}$1) $\neg\neg A \to A$ (D-Antilogism) $\dfrac{((A \wedge B) \to \neg C) \vee D}{((A \wedge C) \to \neg B) \vee D}$

Finally, we write $\Gamma \vdash_c A$ if there is a sequence of formulae $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of $\mathbf{SJ}^{\neg}$; (iii) is obtained by one of the rules (MP)–(D-Antilogism) from formulae preceding it in sequence.

Remark 4.4 As may be verified, the following rule is derivable in $\mathbf{SJ}^{\neg}$:

(Antilogism) $$\frac{(A \wedge B) \to \neg C}{(A \wedge C) \to \neg B}$$

The above axiomatization is inspired by that for classical relevant logic given by Robert Meyer and Richard Routley in [10, p.57].

4.3 Some Basic Results

We first observe two derivable rules in $\mathbf{SJ}^{\neg}$.

Proposition 4.5 *The following formula and rules are derivable in* $\mathbf{SJ}^{\neg}$*.*

(RC¬1) $\dfrac{A \to \neg B}{B \to \neg A}$ (8) $A \to \neg\neg A$ (RC¬2) $\dfrac{A \to B}{\neg B \to \neg A}$

Proof. For (RC¬1), assume $A \to \neg B$. By this, (Ax5) and (Transitivity), we obtain $(B \wedge A) \to \neg B$ and by (Antilogism), we obtain $(B \wedge B) \to \neg A$. Moreover, $B \to (B \wedge B)$ is provable by (Ax1), (Adj), (Ax6) and (MP). Thus we obtain the desired result by (Transitivity). (8) is provable in view of (Ax1) and (RC¬1). For (RC¬2), assume $A \to B$. Then by (8) and (Transitivity), we obtain $A \to \neg\neg B$. Thus, by (RC¬1), we have $\neg B \to \neg A$, as desired. □

Second, we observe that the complete set of de Morgan laws are provable.

Proposition 4.6 *The following formulae are provable in* $\mathbf{SJ}^{\neg}$*.*

(9)	$\neg(A \vee B) \to (\neg A \wedge \neg B)$	(11)	$\neg(A \wedge B) \to (\neg A \vee \neg B)$
(10)	$(\neg A \vee \neg B) \to \neg(A \wedge B)$	(12)	$(\neg A \wedge \neg B) \to \neg(A \vee B)$

Proof. (9) and (10) are easy in view of (RC¬2). For (11), note first that we obtain $\neg(\neg A \vee \neg B) \to (\neg\neg A \wedge \neg\neg B)$ in view of (9), and thus we obtain $\neg(\neg A \vee \neg B) \to (A \wedge B)$ in view of ($\mathrm{Ax}_{\neg}1$). Now, by applying (RC¬2), we obtain $\neg(A \wedge B) \to \neg\neg(\neg A \vee \neg B)$, and this together with ($\mathrm{Ax}_{\neg}1$) and (Transitivity), we obtain the desired result. For (12), note first that we have $A \to (\neg\neg A \vee \neg\neg B)$ and $B \to (\neg\neg A \vee \neg\neg B)$ in view of (8), introduction of disjunction, and (Transitivity). Thus we obtain $(A \vee B) \to (\neg\neg A \vee \neg\neg B)$ by (Adj), (Ax9) and (MP). This together with (10) and (Transitivity) implies $(A \vee B) \to \neg(\neg A \wedge \neg B)$, and finally, by applying (RC¬1), we obtain the desired result. □

Third, we observe that some basic formulae are provable in $\mathbf{SJ}^{\neg}$.

Proposition 4.7 *The following formulae are provable in* $\mathbf{SJ}^{\neg}$*.*

(13) $(A \wedge \neg A) \to B$ (14) $B \to (A \vee \neg A)$

Proof. For (13), by (Ax5), (8) and (Transitivity), we obtain $(A \wedge \neg B) \to \neg\neg A$. By applying (Antilogism), we obtain $(A \wedge \neg A) \to \neg\neg B$, and thus we obtain the desired result in view of (Ax$_\neg$1) and (Transitivity). For (14), by applying (RC$\neg$2) to $(A \wedge \neg A) \to \neg B$, an instance of (13), we obtain $\neg\neg B \to \neg(A \wedge \neg A)$. This together with (8) and (Transitivity) implies $B \to \neg(A \wedge \neg A)$. Moreover, in view of (11) and (Ax$_\neg$1), we obtain $\neg(A \wedge \neg A) \to (A \vee \neg A)$. Thus, the desired result follows by (Transitivity). □

Now we turn to prove the deduction theorem with respect to the material conditional defined in terms of classical negation.

Theorem 4.8 (Classical deduction theorem) *For $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}_\neg$, if $\Gamma, A \vdash_c B$ then $\Gamma \vdash_c \neg A \vee B$.*

Proof. By the induction on the length n of the proof of $\Gamma, A \vdash_c B$. The details are given in the appendix. □

Proposition 4.9 *For $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}_\neg$, if $\Gamma \vdash_c \neg A \vee B$ then $\Gamma, A \vdash_c B$.*

Proof. It suffices to prove the following:

$$(\neg\text{-DS}) \qquad \frac{A \quad \neg A \vee B}{B}.$$

Assume A and $\neg A \vee B$. Then by (Adj), we obtain $A \wedge (\neg A \vee B)$. Thus by (Ax10) and (MP), we have $(A \wedge \neg A) \vee (A \wedge B)$. Now, this together with (13) and (1) implies $B \vee (A \wedge B)$, and thus by making use of (Ax1), (Ax5), (Adj), (Ax9) and (MP), we obtain B, as desired. □

By combining Theorem 4.8 and Proposition 4.9, we obtain the following.

Theorem 4.10 *For $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}_\neg$, we have $\Gamma, A \vdash_c B$ iff $\Gamma \vdash_c \neg A \vee B$.*

4.4 Soundness and completeness

We now proceed to the proof of soundness and completeness.

Theorem 4.11 *For $\Gamma \cup \{A\} \subseteq \mathsf{Form}_\sim$, if $\Gamma \vdash_c A$ then $\Gamma \models_c A$.*

Proof. By induction on the length of the proof, as usual. □

Proposition 4.12 *If $A \vdash_c C$ and $B \vdash_c C$, then $A \vee B \vdash_c C$.*

Proof. Assume $A \vdash_c C$ and $B \vdash_c C$. Then, by Theorem 3.11, we obtain $\vdash_c \neg A \vee C$ and $\vdash_c \neg B \vee C$ respectively. By (Adj), we get $\vdash_c (\neg A \vee C) \wedge (\neg B \vee C)$, and therefore $\vdash_c (\neg A \wedge \neg B) \vee C$ by (2) and (MP). Moreover, by (12) and (1), we obtain $\vdash_c \neg(A \vee B) \vee C$. Finally, we obtain the desired result by another application of Theorem 4.10. □

We are now in a position to prove completeness.

Theorem 4.13 *For $\Gamma \cup \{A\} \subseteq \mathsf{Form}_\neg$, if $\Gamma \models_c A$ then $\Gamma \vdash_c A$.*

Proof. The proof is found in the appendix. □

5 Reflections

5.1 Comparing $\mathbf{SJ}^{\sim}$ and $\mathbf{SJ}^{\neg}$

Although the logics share a lot in common, $\mathbf{SJ}^{\sim}$ and $\mathbf{SJ}^{\neg}$ are incomparable: each has a theorem the other does not, assuming that we normalize the language, assigning the same symbol for both negations, say $\neg$. For instance, we have that $B \to (A \vee {\sim}A)$ is provable in $\mathbf{SJ}^{\neg}$ but not in $\mathbf{SJ}^{\sim}$, while $\neg A \to (\neg\neg A \to B)$ is provable in $\mathbf{SJ}^{\sim}$ but not in $\mathbf{SJ}^{\neg}$.

There are further interesting points of comparison, but due to space constraints, we leave discovery of them to the interested reader.

5.2 Extensions of $\mathbf{SJ}^{\sim}$

One natural extension of $\mathbf{SJ}^{\sim}$ is to full intuitionistic logic with empirical negation, $\mathbf{IPC}^{\sim}$, a system axiomatized and shown strongly complete in [4]. Given the strength of the intuitionistic conditional, the axiomatization of $\mathbf{IPC}^{\sim}$ is much smoother than that of $\mathbf{SJ}^{\sim}$, requiring far fewer rules and a simpler and more familiar axiomatization. We need only add to **IPC** two axioms and one rule governing $\sim$: axioms (Ax$_{\sim}$1) (Law of Excluded Middle) and (Ax$_{\sim}$3) (*ex contradictione quodlibet* for empirically negated formulae) of $\mathbf{SJ}^{\sim}$, and the rule stating that from $A \vee B$, one may infer ${\sim}A \to B$ (called (RP) in [4]).

The most straightforward way to obtain $\mathbf{IPC}^{\sim}$ from $\mathbf{SJ}^{\sim}$ is by adding the axioms corresponding to Heredity and the reflexivity and transitivity of accessibilty, namely,

- $A \to (B \to A)$ (Heredity),
- $(A \to B) \to ((B \to C) \to (A \to C))$ (transitivity),
- $(A \wedge (A \to B)) \to B$ (reflexivity)

The resulting axiomatization is, however, redundant, as a number of the axioms and rules can be shown derivable from a proper subset of the others. [9]

5.3 Extensions of $\mathbf{SJ}^{\neg}$

In [5], Luis Fariñas del Cerro and Andreas Herzig provide an interesting way of adding classical negation to inuititionistic logic without a collapse to classical logic. In this section we relate their results to our own by showing, semantically, that their logic $C+J$ is an extension of the subintuitionistic logic with classical negation $\mathbf{SJ}^{\neg}$. We also strengthen their weak soundness and completeness results to their strong counterparts.

Definition 5.1 A ***CJ**-model* for the language $\mathcal{L}_{\neg}$ is a quadruple $\langle W, g, R, V\rangle$, where W is a non-empty set; $g \in W$ is omniscient; R is a reflexive and transitive relation on W; and $V : W \times \mathsf{Prop} \to \{0,1\}$ an assignment of truth values to state-variable pairs satisfying Heredity. Valuations V are extended to interpretations I to state-formula pairs in the same way as in Definition 4.2.

[9] Note that (Ax10) in $\mathbf{IPC}^{\sim}$ can be swapped with (Ax$_{\sim}$3) of $\mathbf{SJ}^{\sim}$, though this is not the case for $\mathbf{SJ}^{\sim}$ where the order of antecedents matters.

Semantic consequence is defined in terms of truth preservation at g: $\Sigma \models_{\mathbf{CJ}} A$ iff for all models $\langle W, g, R, I\rangle$, $I(g, A) = 1$ if $I(g, B) = 1$ for all $B \in \Sigma$.

Remark 5.2 First, del Cerro and Herzig did not employ the above semantic consequence relation, but instead introduced the notion of validity defined as follows: $\models_{C+J} A$ iff for all models $\langle W, g, R, I\rangle$, $I(w, A) = 1$ for all $w \in W$. Second, while Heredity is ensured for atomic formulae, given the presence of classical negation $\neg$, it fails for arbitrary formulae, in particular, only for formulae containing classical negation.

The following essential notion is employed by del Cerro and Herzig.

Definition 5.3 A formula $A \in \mathsf{Form}_{\neg}$ is *persistent* iff $A \in \mathsf{Prop}$ or A is of the form $B \to C$ for some $B, C \in \mathsf{Form}_{\neg}$.

By making use of this notion, we introduce an extension of $\mathbf{SJ}^{\neg}$.

Definition 5.4 Let **CJ** be the extension of $\mathbf{SJ}^{\neg}$ obtained by adding the following axioms.

$(\mathrm{Ax}_{\mathbf{CJ}}1)$	$A \to (B \to A)$ where A is persistent
$(\mathrm{Ax}_{\mathbf{CJ}}2)$	$(A \wedge (A \to B)) \to B$
$(\mathrm{Ax}_{\mathbf{CJ}}3)$	$(A \to B) \to ((B \to C) \to (A \to C))$

We denote a derivation in **CJ** of A from Γ by $\Gamma \vdash_{\mathbf{CJ}} A$.

Then, we have the following soundness and strong completeness results.

Theorem 5.5 *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}_{\neg}$, $\Gamma \models_{\mathbf{CJ}} A$ *iff* $\Gamma \vdash_{\mathbf{CJ}} A$.

Proof. Ths proof is found in the appendix. □

This leads us to the following identity between $C + J$-validity and **CJ**-validity.

Theorem 5.6 *For* $A \in \mathsf{Form}_{\neg}$, $\models_{C+J} A$ *iff* $\models_{\mathbf{CJ}} A$.

Proof. For the left-to-right direction, it is obvious in view of the definitions of $\models_{\mathbf{CJ}}$ and $\models_{C+J}$. For the other direction, it suffices to prove that if $\vdash_{\mathbf{CJ}} A$ then $\models_{C+J} A$ in light of the previous completeness theorem. This is easily shown by checking that axioms are true at an arbitrary state in the model and that rules preserves validity. We only note that the transitivity of R and Heredity for persistent formulae are used in showing that (DR) preserves validity, and the reflexivity of R is used in showing that (MP) and (DMP) preserve validity. □

6 Final remarks

In neither expansion of **SJ** is the intuitionistic conditional definable from the other connectives. However, if we add the primitive subintuitionistic negation operator $\rightharpoondown$ to either, we can then define the intuitionistic conditional by $A \to B := \rightharpoondown(A \wedge \otimes B)$, for $\otimes \in \{\neg, \sim\}$. Subintuitionistic logic is therefore recoverable as long as we have both negations around (recalling that the falsum is definable in the conditional-free fragments of both $\mathbf{SJ}^{\sim}$ and $\mathbf{SJ}^{\neg}$, in the former by $\sim A \wedge \sim\sim A$ and in the latter by $A \wedge \neg A$). Thus adding classical negation to the conditional-free fragment of intuitionistic logic gives a properly stronger expansion than adding the conditional. Relatedly, adding either subintuitionistic negation or the conditional to classical logic yields the same expansion of classical logic.

We wish to briefly mention some avenues for future research. One is to investigate expansions of relevant logic by empirical negation. Classical relevant logic, recall, is the expansion of the positive fragment $\mathbf{R}^+$ of $\mathbf{R}$ by classical negation, obtained by adding to $\mathbf{R}^+$ the following axioms:

- $\neg\neg A \to A$ (double negation);
- $(A \wedge B) \to \neg C \models (A \wedge C) \to \neg B$ (antilogism).

Both of these classical principles are valid in $\mathbf{SJ}^{\neg}$, but the second fails in $\mathbf{SJ}^{\sim}$, marking an important difference between classical and empirical negation. This failure to validate antilogism makes empirical negation a better fit for relevance logic while still offering a negation of a highly classical nature. The results found here concerning $\mathbf{SJ}^{\sim}$ should carry over to its relevant cousin $\mathbf{B}^+$ expanded by empirical negation, something we hope to explore in future work.

The second avenue for future research concerns the investigation of first-order subintuitionistic logics with classical and empirical negation. Some work in this direction can be found in Ryo Ishigaki and Kentaro Kikuchi's [8].[10] They prove the soundness and weak completeness for Hilbert-style axiom systems by making use of tree-sequent calculi, and the same strategy should be applicable to expansions with classical and empirical negation.

Appendix

Proof of Theorem 2.16

We prove the contrapositive. Suppose that $\Gamma \nvdash A$. By Corollary 2.13, there is a $\Pi \supseteq \Gamma$ such that Π is a prime Π-theory and $A \notin \Pi$. Define the model $\mathfrak{A} = \langle X, \Pi, R, I \rangle$, where $X = \{\Delta : \Delta \text{ is a non-empty non-trivial prime } \Pi\text{-theory}\}$, R defined as in Definition 2.9(v) and I is defined thus. For every state Σ and propositional parameter p:

$$I(\Sigma, p) = 1 \text{ iff } p \in \Sigma$$

We show that this condition holds for an arbitrary formula B:

$$(*) \qquad I(\Sigma, B) = 1 \text{ iff } B \in \Sigma$$

It then follows by a routine induction on the complexity of B that $\mathfrak{A}$ is a counter-model for the inference, and hence that $\Gamma \not\models A$. We show only the interesting case when B has the form $C \to D$.

We have that $I(\Sigma, C \to D) = 1$ iff for all Δ s.t. $\Sigma R \Delta$, if $I(\Delta, C) = 1$ then $I(\Delta, D) = 1$; iff for all Δ s.t. $\Sigma R \Delta$, if $C \in \Delta$ then $D \in \Delta$ by IH; iff $C \to D \in \Sigma$.

[10] See also [16].

For the last equivalence, assume $C \to D \in \Sigma$ and $C \in \Delta$ for any Δ such that $\Sigma R \Delta$. Then by the definition of $\Sigma R \Delta$, we obtain $D \in \Delta$, as desired. On the other hand, suppose $C \to D \notin \Sigma$. Then by Lemma 2.15, there is a Δ such that $\Sigma R \Delta$, $C \in \Delta$, $D \notin \Delta$ and Δ is a prime Π-theory. Furthermore, the non-triviality of Δ follows from the fact that $D \notin \Delta$. Thus, we obtain the desired result.

Proof of Theorem 3.7

By the induction on the length n of the proof of $\Gamma, A \vdash_e B$. If $n = 1$, then we have the following three cases.

- If B is one of the axioms of $\mathbf{SJ}^{\sim}$, then we have $\vdash_e B$. Therefore, by (Ax8), we obtain $\vdash_e {\sim}A \vee B$ which implies the desired result.
- If $B \in \Gamma$, then we have $\Gamma \vdash_e B$, and thus we obtain the desired result by (Ax8).
- If $B = A$, then by (Ax$_{\sim}$1), we have ${\sim}A \vee B$ which implies the desired result.

For $n > 1$, then there are five additional cases to be considered.

- If B is obtained by applying (MP), then we will have $\Gamma, A \vdash_e C$ and $\Gamma, A \vdash_e C \to B$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_e {\sim}A \vee C$ and $\Gamma \vdash_e {\sim}A \vee (C \to B)$, and by (DMP), we obtain $\Gamma \vdash_e {\sim}A \vee B$ as desired.
- If B is obtained by applying (DMP), then $B = D \vee E$ and we will have $\Gamma, A \vdash_e C \vee E$ and $\Gamma, A \vdash_e (C \to D) \vee E$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_e {\sim}A \vee C \vee E$ and $\Gamma \vdash_e {\sim}A \vee (C \to D) \vee E$, and by (DMP), we obtain $\Gamma \vdash_e {\sim}A \vee D \vee E$ as desired.
- If B is obtained by applying (Adj), then $B = C \wedge D$ and we will have $\Gamma, A \vdash_e C$ and $\Gamma, A \vdash_e D$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_e {\sim}A \vee C$ and $\Gamma \vdash_e {\sim}A \vee D$, and by (Adj), (2) and (MP), we obtain $\Gamma \vdash_e {\sim}A \vee (C \wedge D)$ as desired.
- If B is obtained by applying (DR), then $B = ((D \to E) \to (C \to F)) \vee G$ and we will have $\Gamma, A \vdash_e (C \to D) \vee G$ and $\Gamma, A \vdash_e (E \to F) \vee G$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_e {\sim}A \vee (C \to D) \vee G$ and $\Gamma \vdash_e {\sim}A \vee (E \to F) \vee G$, and by (DR), we obtain $\Gamma \vdash_e {\sim}A \vee B$ as desired.
- If B is obtained by applying (DRP), then $B = ({\sim}C \to {\sim}{\sim}D) \vee E$ and we will have $\Gamma, A \vdash_e C \vee D \vee E$ length of the proof of which is less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_e {\sim}A \vee (C \vee D) \vee E$. By (DRP), we obtain $\Gamma \vdash_e {\sim}A \vee ({\sim}C \to {\sim}{\sim}D) \vee E$, i.e. $\Gamma \vdash_e {\sim}A \vee B$ as desired.

This completes the proof.

Proof of Theorem 3.19

We prove the contrapositive. Suppose that $\Gamma \nvdash_e A$. By Corollary 2.13, there is a $\Pi \supseteq \Gamma$ such that Π is a prime Π-theory and $A \notin \Pi$. Define the model $\mathfrak{A} = \langle X, \Pi, R, I \rangle$, where $X = \{\Delta : \Delta$ is a non-empty non-trivial prime Π-theory$\}$, R defined as in Definition 2.9(v) and I is defined thus. For every state Σ and propositional parameter p:

$$I(\Sigma, p) = 1 \text{ iff } p \in \Sigma.$$

We show that this condition holds for an arbitrary formula B:

$$(*) \qquad I(\Sigma, B) = 1 \text{ iff } B \in \Sigma.$$

It then follows that $\mathfrak{A}$ is a counter-model for the inference, and hence that $\Gamma \not\models_e A$. The proof of $(*)$ is by induction on the complexity of B. We show only the case for empirical negation, $\sim$.

We have that $I(\Sigma, {\sim}C) = 1$ iff $I(\Pi, C) \neq 1$; iff $C \notin \Pi$ by IH; iff ${\sim}C \in \Sigma$. For the last equivalence, suppose $C \notin \Pi$ and ${\sim}C \notin \Sigma$ for reductio. Then, by Lemmas 3.16 and 3.18, we obtain ${\sim}C \in \Pi$ and ${\sim}{\sim}C \in \Sigma$. The former and (7) implies that ${\sim}{\sim}C \to D \in \Pi$ since Π is a Π-theory. Moreover, since Σ is also a Π-theory and $\vdash_\Pi {\sim}{\sim}C \to D$ (since ${\sim}{\sim}C \to D \in \Pi_\to$), we obtain $D \in \Sigma$ in view of ${\sim}{\sim}C \in \Sigma$. But D is arbitrary, and this contradicts that Σ is non-trivial. For the other way around, suppose ${\sim}C \in \Sigma$ and $C \in \Pi$ for reductio. Then, by the latter and Lemma 3.17 we obtain ${\sim}{\sim}C \in \Pi$. The rest of the proof is similar to the proof for the other direction, but we use (Ax$_\sim$3) instead of (7).

Proof of Theorem 4.8

By the induction on the length n of the proof of $\Gamma, A \vdash_c B$. If $n = 1$, then we have the following three cases.

- If B is one of the axioms of $\mathbf{SJ}^\neg$, then we have $\vdash_c B$. Therefore, by (Ax8), we obtain $\vdash_c \neg A \vee B$ which implies the desired result.
- If $B \in \Gamma$, then we have $\Gamma \vdash_c B$, and thus we obtain the desired result by (Ax8).
- If $B = A$, then by (14), we have $\neg A \vee B$ which implies the desired result.

For $n > 1$, then there are five additional cases to be considered.

- If B is obtained by applying (MP), then we will have $\Gamma, A \vdash_c C$ and $\Gamma, A \vdash_c C \to B$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_c \neg A \vee C$ and $\Gamma \vdash_c \neg A \vee (C \to B)$, and by (DMP), we obtain $\Gamma \vdash_c \neg A \vee B$ as desired.
- If B is obtained by applying (DMP), then $B = D \vee E$ and we will have $\Gamma, A \vdash_c C \vee E$ and $\Gamma, A \vdash_c (C \to D) \vee E$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_c \neg A \vee C \vee E$ and $\Gamma \vdash_c \neg A \vee (C \to D) \vee E$, and by (DMP), we obtain $\Gamma \vdash_c \neg A \vee D \vee E$ as desired.

- If B is obtained by applying (Adj), then $B = C \wedge D$ and we will have $\Gamma, A \vdash_c C$ and $\Gamma, A \vdash_c D$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_c \neg A \vee C$ and $\Gamma \vdash_c \neg A \vee D$, and by (Adj), (2) and (MP), we obtain $\Gamma \vdash_c \neg A \vee (C \wedge D)$ as desired.
- If B is obtained by applying (DR), then $B = ((D \to E) \to (C \to F)) \vee G$ and we will have $\Gamma, A \vdash_c (C \to D) \vee G$ and $\Gamma, A \vdash_c (E \to F) \vee G$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_c \neg A \vee (C \to D) \vee G$ and $\Gamma \vdash_c \neg A \vee (E \to F) \vee G$, and by (DR), we obtain $\Gamma \vdash_c \neg A \vee B$ as desired.
- If B is obtained by applying (D-Antilogism), then $B = ((C \wedge E) \to \neg D) \vee F$ and we will have $\Gamma, A \vdash_c ((C \wedge D) \to \neg E) \vee F$ length of the proof of which is less than n. Thus, by induction hypothesis, we have $\Gamma \vdash_c \neg A \vee ((C \wedge D) \to \neg E) \vee F$. By (D-Antilogism), we obtain $\Gamma \vdash_c \neg A \vee ((C \wedge E) \to \neg D) \vee F$, i.e. $\Gamma \vdash_c \neg A \vee B$ as desired.

This completes the proof.

Proof of Theorem 4.13

We prove the contrapositive. Suppose that $\Gamma \nvdash_c A$. By Corollary 2.13, there is a $\Pi \supseteq \Gamma$ such that Π is a prime Π-theory and $A \notin \Pi$. Define the interpretation $\mathfrak{A} = \langle X, \Pi, R, I \rangle$, where $X = \{\Delta : \Delta \text{ is a non-trivial prime } \Pi\text{-theory}\}$, R defined as in Definition 2.9(v) and I is defined thus. For every state Σ and propositional parameter p:

$$I(\Sigma, p) = 1 \text{ iff } p \in \Sigma.$$

We show that this condition holds for an arbitrary formula B:

$$(*) \qquad I(\Sigma, B) = 1 \text{ iff } B \in \Sigma.$$

It then follows that $\mathfrak{A}$ is a counter-model for the inference, and hence that $\Gamma \not\models_c A$. The proof of $(*)$ is by induction on the complexity of B. We deal only with classical negation.

We have that $I(\Sigma, \neg C) = 1$ iff $I(\Sigma, C) \neq 1$; iff $C \notin \Sigma$ by IH; iff $\neg C \in \Sigma$. For the last equivalence, assume $C \notin \Sigma$. Since Σ is non-empty, let D be an element of Σ. By (14) we have $\vdash_c D \to (C \vee \neg C)$ and since Σ is a Π-theory, we obtain $C \vee \neg C \in \Sigma$. This together with $C \notin \Sigma$ and the primeness of Σ imply $\neg C \in \Sigma$, as desired. For the other way around, suppose $\neg C \in \Sigma$ and $C \in \Sigma$ for reductio. Then, we obtain $C \wedge \neg C \in \Sigma$ since Σ is a Π-theory. And this together with (13) implies $D \in \Sigma$ for any D. But this contradicts that Σ is non-trivial.

Proof of Theorem 5.5

For soundness, we need to check that the additional axioms are valid. Since ($\text{Ax}_{\mathbf{CJ}}2$) and ($\text{Ax}_{\mathbf{CJ}}3$) are handled by Restall, we check ($\text{Ax}_{\mathbf{CJ}}1$). We split the cases depending on the form of A. For the case when A is a propositional variable, assume, for reductio, $I(g, A \to (B \to A)) \neq 1$. Then, for some

$w_0 \in W$, we have $I(w_0, A) = 1$ and $I(w_0, B \to A) \neq 1$. The latter is equivalent to $I(w_1, B) = 1$ and $I(w_1, A) \neq 1$ for some $w_1 \in W$ such that $w_0 R w_1$. But by Heredity, we also have $I(w_1, A) = 1$ which implies a contradiction. Therefore, we obtain $I(g, A \to (B \to A)) = 1$. For the case when A is of the form $C \to D$, we make use of the transitivity of R. Details are left to the reader.

For completeness, we need to check that the binary relation is reflexive and transitive, and also the persistence condition holds for $p \in \mathsf{Prop}$. Again, we only check the last condition since the other two are shown in [14]. Assume that $p \in \Sigma$ and $\Sigma R \Delta$. Then, since Δ is non-empty, there is an element D such that $D \in \Delta$. And in view of ($\mathrm{Ax}_{\mathbf{CJ}}1$), we have $\vdash_\Pi p \to (D \to p)$, and since Σ is a Π-theory, we obtain $D \to p \in \Sigma$. Finally, by the definition of R and $D \in \Delta$, we obtain $p \in \Delta$, as desired.

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