A Paraconsistent View on B and S5

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Abstract

Paraconsistent logics are logics that in contrast to classical and intuitionistic logic, do not trivialize inconsistent theories. In this paper we show that the famous modal logics **B** and **S5**, can be viewed as paraconsistent logics with several particularly useful properties.

Keywords: KTB, B, modal logic, paraconsistent logic

1 Introduction

One of most counter-intuitive properties of classical logic (as well as of its most famous rival, intuitionistic logic) is the fact that it allows the inference of any proposition from a single pair of contradicting statements. This principle (known as the principle of explosion, 'ex falso sequitur quodlibet') has repeatedly been attacked on philosophical ground, as well as because of practical reasons: in its presence every inconsistent theory or knowledge base is totally trivial, and so useless. Accordingly, over the last decades a lot of work and efforts have been devoted to develop alternatives to classical logic that do not have this drawback. Such alternatives are known as *paraconsistent* logics.

In this paper we embark on a search for a paraconsistent logic which has particularly important properties. The most important of them is what is known as the *replacement* property, which basically means that equivalence of formulas implies their congruence. We show that the minimal paraconsistent logic which satisfies our criteria is in fact the famous Brouwerian modal logic **B** (also known as KTB [13,22]). This logic, in turn, is also shown to be a member of the well-studied family of paraconsistent logics known as C-systems ([16,18,19]). We further show that **B** is very robust paraconsistent logic in the sense that almost any axiom which has been used in the context of C-systems is either a theorem of **B**, or its addition to **B** leads to a logic which is no longer

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paraconsistent. There is exactly one (rather notable) exception, and the result of adding this exception to **B** is another famous modal logic: **S5**.

2 Paraconsistent Logics

We assume that all propositional languages share the same set $\{P_1, P_2, \ldots\}$ of atomic formulas, and use p, q, r to vary over this set. The set of well-formed formulas of a propositional language $\mathcal{L}$ is denoted by $\mathcal{W}(\mathcal{L})$, and φ, ψ, σ will vary over its elements.

Definition 2.1 A (Tarskian) *consequence relation* (tcr) for a language $\mathcal{L}$ is a binary relation $\vdash$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, satisfying the following three conditions:

Reflexivity: if $\psi \in T$ then $T \vdash \psi$.
Monotonicity: if $T \vdash \psi$ and $T \subseteq T'$ then $T' \vdash \psi$.
Transitivity: if $T \vdash \psi$ and $T, \psi \vdash \varphi$ then $T \vdash \varphi$.

Definition 2.2 A *propositional logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a tcr for $\mathcal{L}$ which satisfies the following two conditions:

Structurality: if $T \vdash \varphi$ then $\sigma(T) \vdash \sigma(\varphi)$ for any substitution σ in $\mathcal{L}$.
Non-triviality: $p \nvdash q$ for any distinct propositional variables p, q.

Various general notions of paraconsistent logics have been considered (see, e.g., [14,2,3,1]). In this paper we focus on a particular class of paraconsistent logics, which extend the positive fragment of classical logic as follows.

Notation: $\mathcal{L}_{CL^+} = \{\wedge, \vee, \supset\}$, $\mathcal{L}^{\mathsf{F}}_{CL} = \{\wedge, \vee, \supset, \mathsf{F}\}$ and $\mathcal{L}_{CL} = \{\wedge, \vee, \supset, \neg\}$.

Definition 2.3 $\mathbf{IL}^+$ is the minimal logic $\mathbf{L}$ in $\mathcal{L}_{CL^+}$ such that:

- $\mathcal{T} \vdash_{\mathbf{L}} A \supset B$ iff $\mathcal{T}, A \vdash_{\mathbf{L}} B$
- $\mathcal{T} \vdash_{\mathbf{L}} A \wedge B$ iff $\mathcal{T} \vdash_{\mathbf{L}} A$ and $\mathcal{T} \vdash_{\mathbf{L}} B$
- $\mathcal{T}, A \vee B \vdash_{\mathbf{L}} C$ iff $\mathcal{T}, A \vdash_{\mathbf{L}} C$ and $\mathcal{T}, B \vdash_{\mathbf{L}} C$

$\mathbf{CL}^+$, the $\mathcal{L}_{CL^+}$-fragment of classical logic, is obtained by extending $\mathbf{IL}^+$ with the axiom $A \vee (A \supset B)$. $\mathbf{CL}^{\mathsf{F}}$, the full classical logic (in $\mathcal{L}^{\mathsf{F}}_{CL}$) is obtained by extending $\mathbf{CL}^+$ with the axiom $\mathsf{F} \supset \psi$ (making F a bottom element.) [2]

Definition 2.4 A propositional logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is $\neg$-*classical* if $\mathcal{L}_{CL} \subseteq \mathcal{L}$, the $\mathcal{L}_{CL^+}$-fragment of $\mathbf{L}$ is $\mathbf{CL}^+$, and $\mathbf{L}$ satisfies the three conditions concerning $\vee, \wedge$ and $\supset$ that were used in Definition 2.3 to characterize $\mathbf{IL}^+$.

[2] Another natural alternative for obtaining classical logic is to use the language $\mathcal{L}_{CL}$ rather than $\mathcal{L}^{\mathsf{F}}_{CL}$, and extend $\mathbf{CL}^+$ with the axioms [t] $\neg\psi \vee \psi$ and [$\neg \supset$] $(\neg\psi \supset (\psi \supset \varphi))$. To avoid confusions with the paraconsistent negation which is added to $\mathcal{L}_{CL^+}$ below, we use in this paper the approach with F.

Definition 2.5 A $\neg$-classical logic is *paraconsistent (with respect to $\neg$)* if $\neg$ satisfies the following conditions: (i) $\nvdash_{\mathbf{L}} (p \wedge \neg p) \supset q$, (ii) $\nvdash_{\mathbf{L}} p \supset \neg p$, and (iii) $\nvdash_{\mathbf{L}} \neg p \supset p$.

Remark 2.6 Most of the earlier definitions of paraconsistent logics do not explicitly require conditions (ii) and (iii). However, they have been required in the literature for negation in general (cf. [25,26,29,4,2]. In the latter two papers a connective which satisfies these two conditions is called *weak negation*). In [3,1] paraconsistent logics were defined using an even more restrictive condition (called $\neg$-*containment in classical logic*) on a connective $\neg$ to be counted as a negation in a logic **L**.That condition implies conditions (ii) and (iii) above, but they do suffice for the purposes of this paper.

The above definition mentions only negative properties of negation. Below is a list of positive properties that negation has in classical logic that might be desirable also in the context of paraconsistent logics:

Definition 2.7 Let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a propositional logic for a language $\mathcal{L}$ with a unary connective $\neg$.

- $\neg$ is *complete* (for **L**) if it satisfies the following version of the *law of excluded middle*: (LEM) $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ whenever $\mathcal{T}, \psi \vdash_{\mathbf{L}} \varphi$ and $\mathcal{T}, \neg\psi \vdash_{\mathbf{L}} \varphi$.
- $\neg$ is *right-involutive* (for **L**) if $\varphi \vdash_{\mathbf{L}} \neg\neg\varphi$ every formula φ (equivalently: for atomic φ), and is *left-involutive* (for **L**) if $\neg\neg\varphi \vdash_{\mathbf{L}} \varphi$ for every formula φ (equivalently: for atomic φ). $\neg$ is *involutive* if it is both right- and left-involutive.
- $\neg$ is *contrapositive* (in **L**) if $\neg\varphi \vdash_{\mathbf{L}} \neg\psi$ whenever $\psi \vdash_{\mathbf{L}} \varphi$.

Remark 2.8 It is easy to verify that if **L** is $\neg$-classical then:

- $\neg$ is complete for **L** iff $\vdash_{\mathbf{L}} \neg\varphi \vee \varphi$ for every φ.
- $\neg$ is right-involutive for **L** iff $\vdash_{\mathbf{L}} \varphi \supset \neg\neg\varphi$ for every φ.
- $\neg$ is left-involutive for **L** iff $\vdash_{\mathbf{L}} \neg\neg\varphi \supset \varphi$ for every φ.
- $\neg$ is contrapositive for **L** iff $\vdash_{\mathbf{L}} \neg\varphi \supset \neg\psi$ whenever $\vdash_{\mathbf{L}} \psi \supset \varphi$.

The next proposition shows that $\neg$-classical paraconsistent logics cannot enjoy *all* of the above properties of negation at the same time:

Proposition 2.9 *A $\neg$-classical logic in which $\neg$ is complete, right-involutive, and contrapositive cannot be paraconsistent.*

Completeness (or the law of excluded middle) is a very basic and natural property of negation, which is particularly important to retain in paraconsistent logics (which reject the other basic principle which characterizes classical negation - see Footnote 2). The minimal extension of $\mathbf{CL}^+$ which has complete negation is the paraconsistent logic **CLuN**, introduced by Batens under the name of PI in [6] and further studied in [7,9]; a Hilbert-style system for it is given in Fig. 1.

Demanding just completeness of our paraconsistent negation is obviously not sufficient, though, and we would like to preserve as much of the main classical properties of negation as possible. Proposition 2.9 means that we cannot have a complete paraconsistent negation which is both contrapositive and right-involutive. So we should choose between these two properties. On the other hand the demand of being left-involutive causes no problem, and so we have no reason not to impose it. Doing this leads to the paraconsistent logic known in the literature as $\mathbf{C}_{min}$ (used by the authors in [15,14] as the basis[3] for their taxonomy of C-systems), which is the minimal extension of $\mathbf{CL}^+$ which has both a complete and left-involutive negation. Fig. 1 contains a Hilbert-style system for this logic as well.

Inference Rule: [MP] $\dfrac{\psi \quad \psi \supset \varphi}{\varphi}$

Axioms of HCL^+**:**

[⊃1]	$\psi \supset (\varphi \supset \psi)$
[⊃2]	$(\psi \supset (\varphi \supset \tau)) \supset ((\psi \supset \varphi) \supset (\psi \supset \tau))$
[∧⊃]	$\psi \wedge \varphi \supset \psi, \ \psi \wedge \varphi \supset \varphi$
[⊃∧]	$\psi \supset (\varphi \supset \psi \wedge \varphi)$
[⊃∨]	$\psi \supset \psi \vee \varphi, \ \varphi \supset \psi \vee \varphi$
[∨⊃]	$(\psi \supset \tau) \supset ((\varphi \supset \tau) \supset (\psi \vee \varphi \supset \tau))$
[⊃3]	$((\psi \supset \varphi) \supset \psi) \supset \psi$

Axioms of *HCLuN***:** The axioms of HCL^+ and:

[t]	$\neg\psi \vee \psi$

Axioms of HC_{min}**:** The axioms of *HCLuN* and:

[c]	$\neg\neg\psi \supset \psi$

Fig. 1. The proof systems HCL^+, *HCLuN* and HC_{min}

Returning to the choice between having a negation which is involutive and a negation which is contrapositive, we note that almost no paraconsistent logic studied in the literature has a contrapositive negation[4]. In this paper we investigate what happens if we do follow this choice. As we show below, doing this is more challenging than securing the other properties, as it cannot be achieved by just adding axioms, and so another, more sophisticated way is needed.

[3] In [5] it is argued that the logic **BK**, introduced in the next subsection, is more appropriate as the basic C-system.

[4] An early notable exception is the system CC_ω studied in [38], and extended to a S5-like system in [21]. More recent related works are mentioned in Remark 4.3.

2.1 The family of C-systems

One of the oldest and best known approaches to paraconsistency is da Costa's approach ([16,18,19]), which seeks to allow the use of classical logic whenever it is safe to do so, but behaves completely differently when contradictions are involved. This approach has led to the introduction of the family of *Logics of Formal (In)consistency (LFIs)* ([14,15]). This family is based on the idea that the notion of consistency can be expressed in the language of the logic itself. In most of the LFIs studied in the literature this is done via a *consistency operator*. The expected "classical" behavior of a "consistent" formula ψ for which $\circ\psi$ holds, is expressed via the following conditions:

Definition 2.10 Let $\mathbf{L}$ be a logic for $\mathcal{L}$. A (primitive or defined) connective $\circ$ of $\mathbf{L}$ is a *consistency operator* with respect to $\neg$ if the following conditions are satisfied:

- **(b)** $\vdash_{\mathbf{L}} (\circ\psi \wedge \neg\psi \wedge \psi) \supset \varphi$ for every $\psi, \varphi \in \mathcal{W}(\mathcal{L})$.
- **($\mathbf{n}_1$)** $\not\vdash_{\mathbf{L}} (\circ p \wedge \neg p) \supset q$
- **($\mathbf{n}_2$)** $\not\vdash_{\mathbf{L}} (\circ p \wedge p) \supset q$

We say that $\circ$ is a *strong consistency operator* with respect to $\neg$ if it is a consistency operator which satisfies also (**k**) $\circ\psi \vee (\neg\psi \wedge \psi)$ for every $\psi \in \mathcal{W}(\mathcal{L})$.

Proposition 2.11 *Let* $\mathbf{L}$ *be a* $\neg$*-classical paraconsistent logic.*

- *If* $\circ$ *is a consistency operator with respect to* $\neg$*, then* $\bigotimes\psi =_{Df} (\neg\psi \wedge \psi) \supset \circ\psi$ *is a strong consistency operator with respect to* $\neg$*.*
- *For every* φ*,* $\bigotimes_{\varphi}\psi =_{Df} (\neg\psi \wedge \psi) \supset (\circ\varphi \wedge \neg\varphi \wedge \varphi)$ *is a strong consistency operator.*
- *If* $\circ$ *is a consistency operator while* $\bigotimes$ *is a strong consistency operator, then* $\vdash_{\mathbf{L}} \circ\psi \supset \bigotimes\psi$*.*
- *A strong consistency operator for* $\mathbf{L}$ *is unique up to equivalence.*

Definition 2.12 *Let* $\mathbf{L}$ *be a* $\neg$*-classical logic.* $\mathbf{L}$ *is a* C-system *if it is paraconsistent (w.r.t.* $\neg$*) and has a strong consistency operator* $\circ$ *(w.r.t.* $\neg$*).*

Theorem 2.13 *A* $\neg$*-classical paraconsistent logic is a C-system iff it is an extension (perhaps by definitions) of* $\mathbf{CL}^{\mathsf{F}}$ *(see Definition 2.3).*

The following is what we believe best deserves to be called the basic C-system (the argument is given in [5], see also Footnote 2):

Definition 2.14 The logic **BK** is obtained by extending $\mathbf{CL}^{+}$ with the axioms **(b)** and **(k)**.

Extensions of **BK** with various subsets[5] of the following axioms constitute the main C-systems studied in the literature ($\sharp \in \{\vee, \wedge, \supset\}$):

[5] There are several subsets the addition of which to **BK** results in the loss of paraconsistency, for their full list see [5].

(**c**)	$\neg\neg\varphi \supset \varphi$	(**e**)	$\varphi \supset \neg\neg\varphi$
($\mathbf{n}^{\mathbf{l}}_{\wedge}$)	$\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$	($\mathbf{n}^{\mathbf{r}}_{\wedge}$)	$(\neg\varphi \vee \neg\psi) \supset \neg(\varphi \wedge \psi)$
($\mathbf{n}^{\mathbf{l}}_{\vee}$)	$\neg(\varphi \vee \psi) \supset (\neg\varphi \wedge \neg\psi)$	($\mathbf{n}^{\mathbf{r}}_{\vee}$)	$(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$
($\mathbf{n}^{\mathbf{l}}_{\supset}$)	$\neg(\varphi \supset \psi) \supset (\varphi \wedge \neg\psi)$	($\mathbf{n}^{\mathbf{r}}_{\supset}$)	$(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$
($\mathbf{o}^{\mathbf{1}}_{\sharp}$)	$\circ\varphi \supset \circ(\varphi\sharp\psi)$	($\mathbf{o}^{\mathbf{2}}_{\sharp}$)	$\circ\psi \supset \circ(\varphi\sharp\psi)$
($\mathbf{a}_{\sharp}$)	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi\sharp\psi)$	($\mathbf{a}_{\neg}$)	$\circ\varphi \supset \circ\neg\varphi$
(**l**)	$\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$	(**d**)	$\neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$
($\mathbf{i_1}$)	$\neg\circ\varphi \supset \varphi$	($\mathbf{i_2}$)	$\neg\circ\varphi \supset \neg\varphi$

In what follows we will explore which of the axioms above is already derivable in the logic **NB** studied below, and which can be added to it while preserving its paraconsistency.

3 The Logic NB

3.1 Motivation:

Recall that we set out to find a paraconsistent logic which has a negation which is complete, left-involutive and contrapositive (note that by Proposition 2.9, we cannot also demand right-involutiveness). As we show below, having a contrapositive negation ensures the desirable property (which both classical and intuitionistic logics enjoy) of substitution of equivalents, also known as the replacement property (or self-extensionality [41]):

Definition 3.1 Let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a logic.

- Formulas $\psi, \varphi \in \mathcal{W}(\mathcal{L})$ are *equivalent* in **L**, denoted by $\psi \dashv\vdash_{\mathbf{L}} \varphi$, if $\psi \vdash_{\mathbf{L}} \varphi$ and $\varphi \vdash_{\mathbf{L}} \psi$.
- Formulas $\psi, \varphi \in \mathcal{W}(\mathcal{L})$ are *congruent* (or *indistinguishable*) in **L**, denoted by $\psi \equiv_{\mathbf{L}} \varphi$, if for every formula σ and atom p it holds that $\sigma[\psi/p] \dashv\vdash_{\mathbf{L}} \sigma[\varphi/p]$.
- **L** has the *replacement property* if any two formulas which are equivalent in **L** are congruent in it.

The majority of paraconsistent logics considered in the literature do not have the replacement property. Can we construct ($\neg$-classical) paraconsistent logics which do enjoy the replacement property and have a reasonable (at least complete) negation? The next proposition shows that as long as we want $\neg$ to be complete, this goal cannot be achieved for extensions of **CLuN** by the usual way of adding axioms that force the *strong replacement condition*, where a $\neg$-classical logic **L** satisfies this condition if $\varphi \supset \psi, \psi \supset \varphi \vdash_{\mathbf{L}} \sigma[\psi/p] \supset \sigma[\varphi/p]$ for every atom p and formulas φ, ψ, σ.

Proposition 3.2 *Let* **CAR**[6] *be the logic which is obtained from* **CLuN** *by adding to it the following schema as an axiom:*

$$(\psi \supset \varphi) \wedge (\varphi \supset \psi) \supset (\neg\psi \supset \neg\varphi)$$

[6] It is easy to show that our **CAR** is equivalent to the logic that is called **CAR** in [17]. In Chapter 3 of [35] the same logic (with yet another axiomatization) is called **Le**.

Then **CAR** *is not paraconsistent.*

The above proposition entails that in order to develop paraconsistent extensions of **CLuN** that enjoy the replacement property, the inference of $\neg\varphi \supset \neg\psi$ from $\varphi \supset \psi$ and $\psi \supset \varphi$ should be forced only in the case where the premises are theorems of the logic. This can be done by including this rule in the corresponding proof systems not as a rule of derivation, but just as a *rule of proof*, that is: a rule that is used only to define the set of axioms of the system, but not its consequence relation. To make $\neg$ also contrapositive, it would be better to adopt as a rule of proof the inference of $\neg\varphi \supset \neg\psi$ from $\psi \supset \varphi$ alone. The next proposition implies that as long as we use the language $\mathcal{L}_{CL}$, it would also suffice for forcing the replacement property.

Proposition 3.3 *Let* **L** *be a* $\neg$*-classical logic in* $\mathcal{L}_{CL}$ *which extends* $\mathbf{IL}^+$*, in which* $\vdash_{\mathbf{L}} \neg\varphi \supset \neg\psi$ *whenever* $\vdash_{\mathbf{L}} \psi \supset \varphi$*. Then* **L** *has the replacement property.*

The above considerations lead to the following definition of the logic **NB**:

Definition 3.4 $Th(NB)$ is the minimal set S of formulas in $\mathcal{L}_{CL}$, such that:

(i) S includes all axioms of HC_{min}.

(ii) S is closed under [MP] and the following rule:

$$[CP] \quad \frac{\vdash \psi \supset \varphi}{\vdash \neg\varphi \supset \neg\psi}$$

HNB is the Hilbert-type system whose set of axioms is $Th(NB)$ and has [MP] for $\supset$ as its sole rule of inference.

NB is the logic in $\mathcal{L}_{CL}$ which is induced by HNB.

Obviously, $\vdash_{\mathbf{NB}} \varphi$ iff $\varphi \in Th(NB)$. Note again that [CP] is *not* a rule of inference of HNB, but only a *rule of proof*, i.e., it is used only for defining its set of axioms. This is similar to the role that the necessitation rule (from ψ infer $\Box\psi$) usually has in Hilbert-type systems in modal logics.[7]

The following lemma will be useful in the sequel:

Lemma 3.5

(i) *If* $\vdash_{\mathbf{NB}} \varphi$ *then for every* ψ, $\neg\varphi \vdash_{\mathbf{NB}} \psi$.

(ii) $\vdash_{\mathbf{NB}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$

(iii) $\vdash_{\mathbf{NB}} \neg\neg\neg\varphi \equiv \neg\varphi$ *(that is,* $\vdash_{\mathbf{NB}} \neg\neg\neg\varphi \supset \neg\varphi$ *and* $\vdash_{\mathbf{NB}} \neg\varphi \supset \neg\neg\neg\varphi$*).*

Remark 3.6 The first item of Lemma 3.5 implies that if we take F to be an abbreviation of $\neg(P_1 \supset P_1)$ (say), then for every φ, $\vdash_{\mathbf{NB}} \mathsf{F} \supset \varphi$. Hence we

[7] More precisely: whether necessitation is taken in modal logics as a rule of proof or a rule of derivation depends on the intended consequence relation. If the *local* one of preserving truth in worlds is used, then the rule can be taken only as a rule of proof. In contrast, if the *global* one of preserving validity in frames (that is, truth in all worlds of a frame) is used, then the rule should be taken as a rule of derivation.

may assume that the language of **NB** is an extension of $\mathcal{L}_{CL}^{\mathsf{F}}$, and that every instance of a classical tautology in $\mathcal{L}_{CL}^{\mathsf{F}}$ is in $Th(NB)$.

Next we define a Gentzen-style system for **NB**.

Notation: $\neg S = \{\neg\varphi \mid \varphi \in S\}$.

Definition 3.7 The system GNB is obtained from the Gentzen-style system LK for **CL** ([20]) by replacing its left introduction rule for negation ($[\neg\Rightarrow]$) by the rule:

$$[\neg\Rightarrow]_B \qquad \frac{\Gamma, \neg\Delta \Rightarrow \psi}{\neg\psi \Rightarrow \neg\Gamma, \Delta}$$

Theorem 3.8 *$\mathcal{T} \vdash_{GNB} \varphi$ (i.e., there is a finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow \varphi$ is derivable in GNB) iff $\mathcal{T} \vdash_{\mathbf{NB}} \varphi$.*

Proposition 3.9 *GNB does not admit cut-elimination.*

Proof. Obviously, $\vdash_{GNB} \neg(p\vee q), \neg(p\vee q) \to r \Rightarrow r$. By applying $[\neg\Rightarrow]_B$ we get from this that $\vdash_{GNB} \neg r \Rightarrow \neg(\neg(p\vee q) \to r), p\vee q$. Since also $\vdash_{GNB} p\vee q \Rightarrow p, q$, an application of the Cut rule yields that $\vdash_{GNB} \neg r \Rightarrow \neg(\neg(p \vee q) \to r), p, q$. On the other hand a straightforward (though tedious) search reveals that this sequent has no cut-free proof in GNB. □

In the next subsection we will see that GNB does admit a weaker version of cut-elimination and this suffices for making it a decidable system which has the crucial subformula property.

3.2 Kripke-style Semantics for NB

For providing adequate semantics for **NB**, we use the following framework of Kripke frames for modal logics.

Definition 3.10 A triple $\langle W, R, \nu\rangle$ is called a **NB**-*frame* for $\mathcal{L}_{CL}$ [8], if W is a nonempty (finite) set (of "worlds"), R is a reflexive and symmetric relation on W, and $\nu : W \times \mathcal{W}(\mathcal{L}_{CL}) \to \{t, f\}$ satisfies the following conditions:

- $\nu(w, \psi \wedge \varphi) = t$ iff $\nu(w, \psi) = t$ and $\nu(w, \varphi) = t$.
- $\nu(w, \psi \vee \varphi) = t$ iff $\nu(w, \psi) = t$ or $\nu(w, \varphi) = t$.
- $\nu(w, \psi \supset \varphi) = t$ iff $\nu(w, \psi) = f$ or $\nu(w, \varphi) = t$.
- $\nu(w, \neg\psi) = t$ iff there exists $w' \in W$ such that wRw', and $\nu(w', \psi) = f$.

Definition 3.11 Let $\langle W, R, \nu\rangle$ be a **NB**-frame.

- A formula φ is *true* in a world $w \in W$ ($w \Vdash \varphi$) if $\nu(w, \varphi) = t$.
- A sequent $s = \Gamma \Rightarrow \Delta$ is *true* in a world $w \in W$ ($w \Vdash s$) if $\nu(w, \varphi) = f$ for some $\varphi \in \Gamma$, or $\nu(w, \varphi) = t$ for some $\varphi \in \Delta$. Equivalently, $w \Vdash s$ if $w \Vdash I(s)$,

[8] In the literature on modal logics one usually means by a "frame" just the pair $\langle W, R\rangle$, while we find it convenient to follow [34], and use this technical term a little bit differently, so that the valuation ν is a part of it.

where $I(s)$ is the usual interpretation of s (as defined, e.g., in the proof of Theorem 3.8).

- A formula φ is *valid* in $\langle W, R, \nu\rangle$ ($\langle W, R, \nu\rangle \models \varphi$) if it is true in every world $w \in W$.
- A sequent s is *valid* in $\langle W, R, \nu\rangle$ ($\langle W, R, \nu\rangle \models s$) if it is true in every world $w \in W$.

Definition 3.12

- Let $\mathcal{T} \cup \{\varphi\}$ be a set of formulas in $\mathcal{L}_{CL}$. φ *semantically follows in* **NB** from $\mathcal{T}$ if for every **NB**-frame $\langle W, R, \nu\rangle$ and every $w \in W$: if $w \Vdash \psi$ for every $\psi \in \mathcal{T}$ then $w \Vdash \varphi$.
- Let $S \cup \{s\}$ be a set of sequents in $\mathcal{L}_{CL}$. s *semantically follows in* **NB** from S if for every **NB**-frame $\mathcal{W}$, if $\mathcal{W} \models s'$ for every $s' \in S$, then $\mathcal{W} \models s$. s is **NB**-*valid* if s semantically follows in **NB** from $\emptyset$ (that is, s is valid in every **NB**-frame).

Proposition 3.13

(i) *If φ is a theorem of* **NB** *(that is, $\varphi \in Th(NB)$), then φ is valid in every* **NB**-*frame.*

(ii) *If $\mathcal{T} \vdash_{\mathbf{NB}} \varphi$ then φ semantically follows in* **NB** *from $\mathcal{T}$.*

(iii) *Let $S \cup \{s\}$ be a set of sequents. If $S \vdash_{GNB} s$ then s semantically follows in* **NB** *from S. In particular: if $\vdash_{GNB} s$ then s is* **NB**-*valid.*

Now we turn to prove the completeness of **NB** for its possible-worlds semantics, as well as the analyticity of GNB. The latter property is defined as follows:

Definition 3.14 Let G be a Gentzen-type system in a language $\mathcal{L}$.

- Let $\mathcal{F}$ be a set of formulas in $\mathcal{L}$. A proof in G is called $\mathcal{F}$-*analytic* if every formula which occurs in it belongs to $\mathcal{F}$.
- Let $S \cup \{s\}$ be a set of sequents in $\mathcal{L}$. A proof in G of s from S is called *analytic* if it is $\mathcal{F}$-analytic, where $\mathcal{F}$ is the set of subformulas of formulas in $S \cup \{s\}$.
- G has the *(strong) subformula property* if whenever $\vdash_{\mathsf{G}} s$ ($S \vdash_{\mathsf{G}} s$), there is an analytic proof of s (from S).

Theorem 3.15 *Let $S \cup \{s\}$ be a finite set of sequents in $\mathcal{L}_{CL}$. If s semantically follows in* **NB** *from S then s has an analytic proof in GNB from S.*

Proof. Suppose s does not have an analytic proof in GNB from S. We construct a **NB**-frame in which the elements of S are valid, but s is not.

Denote by $\mathcal{F}$ the set of subformulas of formulas in $S \cup \{s\}$. Call a sequent $\Gamma \Rightarrow \Delta$ $\mathcal{F}$-*maximal* if the following conditions (i) $\Gamma \cup \Delta = \mathcal{F}$, and (ii) $\Gamma \Rightarrow \Delta$ has no $\mathcal{F}$-analytic proof from S.

Lemma 1. Suppose $\Gamma \cup \Delta \subseteq \mathcal{F}$, and $\Gamma \Rightarrow \Delta$ has no $\mathcal{F}$-analytic proof from

S. Then $\Gamma \Rightarrow \Delta$ can be extended to an $\mathcal{F}$-maximal sequent $\Gamma' \Rightarrow \Delta'$ (that is, $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$).

Proof of Lemma 1. Let $\Gamma' \Rightarrow \Delta'$ be a maximal extension of $\Gamma \Rightarrow \Delta$ that consists of formulas in $\mathcal{F}$, and has no $\mathcal{F}$-analytic proof from S. (Such $\Gamma' \Rightarrow \Delta'$ exists, because $\mathcal{F}$ is finite.) To show that $\Gamma' \Rightarrow \Delta'$ is $\mathcal{F}$-maximal, assume for contradiction that there is $\varphi \in \mathcal{F}$ such that $\varphi \notin \Gamma' \cup \Delta'$. Then the maximality of $\Gamma' \Rightarrow \Delta'$ implies that both $\Gamma' \Rightarrow \Delta', \varphi$ and $\varphi, \Gamma' \Rightarrow \Delta'$ have $\mathcal{F}$-analytic proofs from S. But then we can get an $\mathcal{F}$-analytic proof from S using these two proofs together with an application of a cut on φ to their conclusions. (Note that since $\varphi \in \mathcal{F}$, the resulting proof of $\Gamma' \Rightarrow \Delta'$ is still $\mathcal{F}$-analytic.) This contradicts our assumption about $\Gamma' \Rightarrow \Delta'$.

Since s has no $\mathcal{F}$-analytic proof from S, it follows from Lemma 1 that s can be extended to an $\mathcal{F}$-maximal sequent $\Gamma^* \Rightarrow \Delta^*$.

Let W be the set of all $\mathcal{F}$-maximal sequents. Since $\mathcal{F}$ is finite, so is W. Since $(\Gamma^* \Rightarrow \Delta^*) \in W$, W is also nonempty. Define a relation R on W as follows: $(\Gamma_1 \Rightarrow \Delta_1)R(\Gamma_2 \Rightarrow \Delta_2)$ iff for every formula φ, if $\neg\varphi \in \Delta_1$ then $\varphi \in \Gamma_2$, and if $\neg\varphi \in \Delta_2$ then $\varphi \in \Gamma_1$. Obviously, R is symmetric. That it is also reflexive follows from the fact that if $\{\neg\varphi, \varphi\} \subseteq \Delta$ then $\Gamma \Rightarrow \Delta$ has a cut-free proof in GNB (since it can be derived from the axiom $\varphi \Rightarrow \varphi$ using $[\Rightarrow\neg]$ and weakenings). This fact and the $\mathcal{F}$-maximality of a sequent $\Gamma \Rightarrow \Delta$ in W imply that if $\neg\varphi \in \Delta$ then $\varphi \in \Gamma$, and so $(\Gamma \Rightarrow \Delta)R(\Gamma \Rightarrow \Delta)$. Next, let $\mathcal{W}$ be the **NB**-frame $\langle W, R, \nu\rangle$ in which W and R are as above, and ν is obtained by letting $\nu(\Gamma \Rightarrow \Delta, p) = t$ iff $p \in \Gamma$ (p atomic).

Lemma 2. Let $\Gamma \Rightarrow \Delta \in W$ and $\varphi \in \mathcal{F}$. Then $\nu(\Gamma \Rightarrow \Delta, \varphi) = t$ if $\varphi \in \Gamma$, and $\nu(\Gamma \Rightarrow \Delta, \varphi) = f$ if $\varphi \in \Delta$.

Proof of Lemma 2. By induction on the complexity of φ.

Since $\Gamma^* \Rightarrow \Delta^*$ is an extension of s, it follows from Lemma 2 that if $s = \Gamma \Rightarrow \Delta$, then $\nu(\Gamma^* \Rightarrow \Delta^*, \varphi) = t$ for every $\varphi \in \Gamma$, while $\nu(\Gamma^* \Rightarrow \Delta^*, \varphi) = f$ for every $\varphi \in \Delta$. It follows that $\Gamma^* \Rightarrow \Delta^* \not\Vdash s$, and so $\mathcal{W} \not\Vdash s$.

Finally, let $s' = (\Gamma' \Rightarrow \Delta') \in S$, and let $w = (\Gamma \Rightarrow \Delta) \in W$. It is impossible that w is an extension of s', because w has no $\mathcal{F}$-analytic proof from S. It follows that either $\varphi \in \Delta$ for some $\varphi \in \Gamma'$, or $\varphi \in \Gamma$ for some $\varphi \in \Delta'$. By Lemma 2 this implies that either $\nu(w, \varphi) = f$ for some $\varphi \in \Gamma'$, or $\nu(w, \varphi) = t$ for some $\varphi \in \Delta'$. Hence $w \Vdash s'$ for every $w \in W$, and so $\mathcal{W} \Vdash s'$ for every $s' \in S$. □

Corollary 3.16 *GNB has the subformula property: if $S \vdash_{GNB} s$ then s has an analytic proof in GNB from S. In particular: if $\vdash_{GNB} s$ then s has an analytic proof in GNB.*[9]

Corollary 3.17 *If Γ is finite then $\Gamma \vdash_{\mathbf{NB}} \varphi$ iff φ semantically follows in* **NB** *from Γ.*

[9] This implies that if $\vdash_{GNB} s$ then s has a proof in GNB in which all cuts are analytic (that is, the cut formulas are subformulas of s).

The above can be strengthened to full completeness (we leave the proof to the reader):

Theorem 3.18 *For every theory $\mathcal{T}$, $\mathcal{T} \vdash_{\mathbf{NB}} \varphi$ iff φ semantically follows in* **NB** *from $\mathcal{T}$.*

Remark 3.19 *GNB* is a version of the Gentzen-type system for **B** given in [39] (and described in [40]). Unlike our proof, the analyticity of the assumptions-free fragment of that system is proved in [39] by syntactic [10] means.

3.3 Basic Properties of NB

From Proposition 2.9 and Note 2.8 it follows that a logic which has a complete, left-involutive and contrapositive negation should contain the logic **NB**. The next proposition shows that **NB** is in fact the minimal logic of this type:

Proposition 3.20 **NB** *is the minimal extension of* $\mathbf{CL}^+$ *in* $\mathcal{L}_{CL}$ *in which* $\neg$ *is complete, contrapositive, and left-involutive.*

Proposition 3.21 **NB** *is paraconsistent and has the replacement property.*

Theorem 3.22 **NB** *is decidable.*

Proof. Given a formula φ (or a finite set of formulas $\Gamma \cup \{\varphi\}$), the number of sequents which consist only of subformulas of φ (or $\Gamma \Rightarrow \varphi$) is finite. Hence it easily follows [11] from Theorem 3.16 that it is decidable whether $\vdash_{GNB} \varphi$ (or $\vdash_{GNB} \Gamma \Rightarrow \varphi$) or not. □

3.4 NB as a C-system

From Note 3.6 it follows that we may assume (as we do from this point on) that the language of **NB** includes F, and that **NB** is an extension of $\mathbf{CL}^{\mathsf{F}}$. Therefore Theorem 2.13 implies that **NB** (and any of its paraconsistent extensions) is a *C*-system. From Proposition 2.11, the replacement property of **NB**, and the first item of Lemma 3.5 it further follows that (any paraconsistent extension in $\mathcal{L}_{CL}$ of) **NB** has a *unique* (up to congruence) strong consistency operator $\circ$, which can be defined as $\circ\varphi =_{def} \varphi \wedge \neg\varphi \supset \mathsf{F}$, or as $\circ\varphi =_{def} (\varphi \wedge \neg\varphi) \supset \neg(\varphi \supset \varphi)$.

We now check which of the schemas listed in subsection 2.1 is valid in **NB**, and which can be added to it without losing its paraconsistency.

Proposition 3.23 *The following schemas from the list given in subsection 2.1 are provable in* **NB** *(in addition to* (**b**), (**k**), (**t**) *and* (**c**)*):* $(\mathbf{n}^{\mathbf{l}}_{\wedge})$, $(\mathbf{n}^{\mathbf{r}}_{\wedge})$, $(\mathbf{n}^{\mathbf{l}}_{\vee})$, $(\mathbf{n}^{\mathbf{1,2}}_{\supset})$, $(\mathbf{a}_{\neg})$, $(\mathbf{a}_{\wedge})$, *and* $(\mathbf{a}_{\vee})$.

Remark 3.24 The fact that $(\mathbf{a}_{\neg})$, $(\mathbf{a}_{\wedge})$, and $(\mathbf{a}_{\vee})$ are all valid in **NB** means that **NB** is almost perfectly adequate to serve as a C-system according to da

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[10] A semantic proof appeared in [23] (Example 5.54), as a particular instance of a general method for proving analyticity.

[11] Instead of using the Gentzen-type system *GNB*, one can use the semantics of **NB** in order to provide a decision procedure for it. This is due to the fact that from the proof of Theorem 3.15 it follows that a sequent s is **NB**-valid iff it is valid in every **NB**-frame in which the number of worlds is at most 2^n, where n is the number of subformulas of s.

Costa's ideas. The only principle that it misses (as we show is the next theorem) is $(\mathbf{a}_\supset)$. This is the price it pays for being contrapositive and for having the replacement property. However, this is not a high price, since the language of $\{\neg, \wedge, \vee\}$ suffices for classical reasoning (since its set of primitive connectives is functionally complete for two-valued matrices).

It is also remarkable that $\neg(\varphi \wedge \psi)$ is equivalent (and so congruent) in **NB** to $\neg\varphi \vee \neg\psi$. However, the next theorem shows that the other De Morgan rules are only partially valid in **NB**. Another important fact that is shown in the next theorem is that with one exception (to be dealt with in the sequel), all the schemas from the list in Subsection 2.1 that are not already derivable in **NB** cannot even be added to it without losing its paraconsistency. This shows that **NB** is rather robust as a paraconsistent logic.

Theorem 3.25 *Let* **L** *be obtained by adding to HNB as an axiom any element of the set* $\{(\mathbf{e}), (\mathbf{n}^{\mathbf{l},\mathbf{1}}_\supset), (\mathbf{n}^{\mathbf{r}}_\vee), (\mathbf{n}^{\mathbf{r}}_\supset), (\mathbf{a}_\supset), (\mathbf{i_1}), (\mathbf{l}), (\mathbf{d})\}$*, or any axiom of the form* $(\mathbf{o}^{\mathbf{i}}_\#)$ *(*$\sharp \in \{\wedge, \vee, \supset\}$*,* $i \in \{1, 2\}$*). Then* **L** *is not -paraconsistent.*

Proof. We show for the cases of $(\mathbf{e})$ and $(\mathbf{n}^{\mathbf{l},\mathbf{1}}_\supset)$, leaving the rest of the cases to the reader.

- Suppose $(\mathbf{e})$ is valid in **L**. Then from Proposition 2.9 it follows that $\neg\varphi \supset (\varphi \supset \neg\neg\psi)$ for every φ, ψ. Since $(\mathbf{c})$ is valid in **NB**, this implies that $\vdash_{\mathbf{L}} \neg\varphi \supset (\varphi \supset \psi)$.
- Suppose $(\mathbf{n}^{\mathbf{l},\mathbf{1}}_\supset)$ is valid in **L**. Then $\vdash_{\mathbf{L}} \neg(\varphi \supset \psi) \supset \varphi$. By applying [CP] we get that $\vdash_{\mathbf{L}} \neg\varphi \supset \neg\neg(\varphi \supset \psi)$. Hence $\vdash_{\mathbf{L}} \neg\varphi \supset (\varphi \supset \psi)$.

$\square$

Remark 3.26 The exception which has not been dealt with in Proposition 3.23 and Theorem 3.25 is the schema $(\mathbf{i_2})$. Now it is not difficult to show that $(\mathbf{i_2})$ is not provable in **NB**. In the sequel we show that it can nevertheless be added to **NB** without losing its paraconsistency, and that this addition leads to another interesting logic.

3.5 NB is the Modal Logic B

The notion of '**NB**-frame' is very similar to the notion of a Kripke frame used in the study of modal logics. Indeed, **NB** is actually (equivalent to) the famous modal logic which is usually called **B** or **KTB** (see e.g. [13]). The language of **B** is usually taken to be $\{\wedge, \vee, \supset, \mathsf{F}, \Box\}$ (or $\{\wedge, \vee, \supset, \neg, \Box\}$, where $\neg$ denotes the *classical* negation). Its semantics is given by Kripke frames in which the accessibility relation R is again reflexive and symmetric, where the notion of a 'Kripke frame' is defined like in Defn 3.10, except that instead of the clause there for $\neg$ we have the following clause for $\Box$:

- $\nu(w, \Box\psi) = t$ iff $\nu(w', \psi) = t$ for every $w' \in W$ such that wRw'.

Now it is easy to see that with respect to Kripke frames, the language of our **NB** and the language of the modal logic **B** are equivalent in their expressive power. $\Box$ is definable in the former by $\Box\varphi =_{def} \sim\neg\varphi$, where $\sim \psi =_{def} \psi \supset \mathsf{F}$. On the other hand $\neg$ is definable in the language of **B** by $\neg\varphi =_{def} \sim \Box\varphi$. It follows that the paraconsistent logic **NB** (whose language is just $\mathcal{L}_{CL}$) and the modal logic **B** are practically identical.

It is worth noting that the presentation of the modal **B** in the form **NB** is more concise (and in our opinion also clearer) than the usual one in two ways. First, **NB** really has only two basic connectives: $\supset$ and $\neg$. (F can be defined as $\neg(\varphi \supset \varphi)$, where φ is arbitrary, and $\vee$ and $\wedge$ can of course be defined in terms of $\supset$ and F.) The standard presentation of **B** needs three connectives: $\supset$, F, and $\Box$. Second, the standard Hilbert-type proof system for **B** is more complicated than HNB. It is obtained from (the full) HCL by the addition of one rule of proof and three axioms. The rule is the necessitation rule (if $\vdash \varphi$ then $\vdash \Box\varphi$). The three axioms are: **(K)** $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$, **(T)** $\Box\varphi \supset \varphi$ and **(B)** $\varphi \supset \Box\Diamond\varphi$, where $\Diamond\varphi =_{def} \sim\Box\sim\varphi$. In contrast, HNB is obtained from HCL^+ by the addition of one rule of proof (which admittedly is somewhat more complex than the necessitation rule), and just two extremely simple and natural axioms.

4 The Logic NS5

Following Note 3.26, in this section we investigate the system that is obtained from **NB** by the addition of $(\mathbf{i_2})$. We start by presenting two schemas which are equivalent to $(\mathbf{i_2})$ over **NB**.

Lemma 4.1 *The logics which are obtained by extending* **NB** *with one of the following schemas are identical.*

(i) $(\mathbf{i_2})$ *(that is:* $\neg\circ\varphi \supset \neg\varphi$*).*

(ii) $\circ(\neg\varphi)$ *(that is:* $\neg\varphi \wedge \neg\neg\varphi \supset \mathsf{F}$*).*

(iii) $\neg\neg\varphi \supset (\neg\varphi \supset \psi)$.

Definition 4.2 Let $HNS5$ be the Hilbert-type system which is obtained from HNB by adding to it as an axiom schema one of the three schemas which were proved equivalent in Lemma 4.1. **NS5** is the logic induced by $HNS5$.

Remark 4.3 Béziau ([10,11,12] and Batens ([8]) have introduced systems equivalent to **NS5**. (**NS5** was called $\mathbb{Z}$ by Béziau, and **A** by Batens.) Further study of this system was done in [37]. The Hilbert-type system $HNS5$ presented above is an improved version of the Hilbert-type system for $\mathbb{Z}$ presented in that paper. The same simplified axiomatization of HNS5 was also independently discovered by Omori and Waragai and presented in [36]. Actually, our axiomatization HNB of the modal logic **KTB** is implicitly given there as well. [12] At this point it is worth noting also that the realization on which the present paper is based (that the same method that was applied to **S5** can be applied to other modal logics in order to produce interesting paraconsistent logics) was first pursued independently in [28,27,30] and in [31,32,33]. Another investigation of paraconsistent logics from a modal viewpoint, studying also

[12] We are grateful to an anonymous referee for bringing this paper and these facts to our attention.

analytic and cut-free sequent calculi for such logics, is presented in the current volume ([24]).

Remark 4.4 The following observation leads to a simpler version of $HNS5$:

$$\neg\neg\varphi \supset (\neg\varphi \supset \varphi), \neg\varphi \vee \varphi \vdash_{\mathbf{IL}^+} \neg\neg\varphi \supset \varphi$$

It follows that in order to axiomatize **NS5**, it suffices to add to HCL^+ the schemas $\neg\varphi \vee \varphi$, $\neg\neg\varphi \supset (\neg\varphi \supset \psi)$, and the rule [CP] (or to add to **CLuN** the schema $\neg\neg\varphi \supset (\neg\varphi \supset \psi)$, and the rule [CP]).

Like in the case of **NB**, we provide a Gentzen-type system and Kripke-style semantics for it, leaving all proofs in the section to the reader.

Definition 4.5 [$GNS5$] The system $GNS5$ is the system which is obtained from LK by replacing its rule $[\neg\Rightarrow]$ by the rule:

$$[\neg\Rightarrow]_5 \quad \frac{\neg\Gamma \Rightarrow \psi, \neg\Delta}{\neg\Gamma, \neg\psi \Rightarrow \neg\Delta}$$

Theorem 4.6 *$\mathcal{T} \vdash_{GNS5} \varphi$ iff $\mathcal{T} \vdash_{\mathbf{NS5}} \varphi$.*

Definition 4.7

- An **NB**-frame $\langle W, R, \nu\rangle$ is called a **NS5**-*frame* for $\mathcal{L}_{CL}$ if R is transitive (in addition to its being reflexive and symmetric).
- The notions of truth (in worlds) and validity in **NS5**-frames (of formulas and sequents) are defined like in Definition 3.11.
- Semantic consequence in **NS5** is defined like in Definition 3.12, using **NS5**-frames instead of **NB**-frames.

Proposition 4.8

(i) *If φ is a theorem of* **NS5** *then φ is valid in every* **NS5**-*frame.*

(ii) *If $\mathcal{T} \vdash_{\mathbf{NS5}} \varphi$ then φ semantically follows in* **NS5** *from $\mathcal{T}$.*

(iii) *Let $S \cup \{s\}$ be a set of sequents. If $S \vdash_{GNS5} s$ then s semantically follows in* **NS5** *from S. In particular: if $\vdash_{GNS5} s$ then s is* **NS5**-*valid.*

Theorem 4.9 *Let $S \cup \{s\}$ be a finite set of sequents in $\mathcal{L}_{CL}$. If s semantically follows in* **NS5** *from S then s has an analytic proof in $GNS5$ from S.*

Theorem 4.9 has for **NS5** the same important corollaries as Theorem 3.15 has for **NB**.

Theorem 4.10 *$GNS5$ has the subformula property: if $S \vdash_{GNS} s$ then s has an analytic proof in $GNS5$ from S. In particular: if $\vdash_{GNS5} s$ then s has an analytic proof in $GNS5$.*

Theorem 4.11 *For finite Γ, $\Gamma \vdash_{\mathbf{NS5}} \varphi$ iff φ semantically follows in* **NS5** *from Γ.*

Theorem 4.12 **NS5** *is decidable.*

Proposition 4.13 **NS5** *is a* $\neg$*-classical paraconsistent logic with a complete, contrapositive, and left-involutive negation. It is also a C-system in which all the schemas listed in Proposition 3.23 are valid, as well as* $(\mathbf{i_2})$ *and* $\circ\neg\varphi$.

Theorem 4.14 **NS5** *is equivalent to the famous modal logic* **S5** *(also known as* **KT5** *or* **KT45***).*

Remark 4.15 The modal logic **S5** is the logic induced by the class of Kripke frames in which the accessibility relation is an equivalence relation. Theorem 4.14 follows from this characterization of **S5**. Note that the standard Hilbert-type system for **S5** is obtained from that of **B** by replacing the axiom (B) by the axiom **(5)** $\diamond\varphi \supset \Box\diamond\varphi$. It is worth noting also that in **NS5** $\Box\varphi$ (that is: $\sim\neg\varphi$) is equivalent to $\neg\neg\varphi$. (This can easily be shown by using the semantics, or by using $GNS5$.)

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