Neighbourhood Frame Product KxK

Andrey Kudinov [1]

Institute for Information Transmission Problems, RAS, Moscow, Russia,
National Research University Higher School of Economics, Moscow, Russia
Moscow Institute of Physics and Technology

Abstract

We consider modal logics of products of neighborhood frames and find the modal logic of all products of normal neighborhood frames.

Keywords: Modal logic, neighborhood frame, product of modal logics, product neighborhood frame.

1 Introduction

In this paper we continue the research of [5] and study modal logics of products of neighborhood frames.

Neighborhood frames, as a generalization of Kripke semantics for modal logic, were introduced independently by Dana Scott [9] and Richard Montague [7]. Neighborhood semantics is more general than Kripke semantics, and in case of normal reflexive and transitive logics, coincides with topological semantics. In this paper we consider the product of neighborhood frames introduced by Sano in [8]. It is a generalization of the product of topological spaces [2] presented in [1].

The product of neighborhood frames is defined in the vein of the product of Kripke frames (see [11] and [12]). But there are some differences. Axioms of commutativity and Church-Rosser property are valid in any product of Kripke frames. Whereas in [1] it was shown that the logic of the products of all topological spaces is the fusion of logics $\mathsf{S4} * \mathsf{S4}$. Even more, $\mathsf{S4} * \mathsf{S4}$ is complete w.r.t. the product $\mathbb{Q} \times_t \mathbb{Q}$ ($\times_t$ stands for product of topological spaces, defined in [1]).

In [5] this result was extended. It was proven that for any pair L and L' of logics from set $\{\mathsf{S4}, \mathsf{D4}, \mathsf{D}, \mathsf{T}\}$ modal logic of products of L-neighborhood frames and L'-neighborhood frames is the fusion of L and L'. But it was unclear how to proceed in case of logics that do not contain axiom $\Diamond\top$ (correspond to seriality). In this paper we show that any product of neighborhood frames in

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[2] "Product of topological spaces" is a well-known notion in Topology but it is different from what we use here (for details see [1]).

fact satisfy axiom $\psi \to \Box_2\psi$, where ψ is a variable-free and $\Box_2$-free formula (and similar for $\Box_1$). We prove that $\mathsf{K} * \mathsf{K}$ plus all such axioms will be the logic of all products of neighborhood frames.

Neighborhood frames are often considered in the context of non-normal modal logics, since, unlike Kripke semantics, it is complete w.r.t. many non-normal logics. As for normal modal logic, neighborhood frames rarely gives anything new in comparison to Kripke frames. This paper, however, shows that normal neighborhood frames, that correspond to normal modal logics, give different results than Kripke frames in case of products.

The results of this paper (and others: [1], [5], [8]) show that "neighborhood" product, in general, gives weaker logic than "Kripke" product of modal logics. From this we can conclude that neighborhood semantics is a finer tool for products of modal logics even for normal modal logics. It also shows that notion of the product of modal logics depend on the semantics.

We should also mention the possibility of adding the third modality that, in topological context, correspond to classical product topology. That was done in [1] for $\mathsf{S4}$ and topological semantics with the interior operator and in [6] for $\mathsf{D4}$ and topological semantics with the derivational operator. It may be possible to consider similar construction for other logics.

2 Language and logics

In this paper we study propositional modal logics. A formula is defined recursively as follows:

$$\phi ::= p \mid \bot \mid \phi \to \phi \mid \Box_i\phi,$$

where $p \in \mathrm{PROP}$ is a propositional letter and $\Box_i$ is a modal operator. Other connectives are introduced as abbreviations: classical connectives are expressed through $\bot$ and $\to$, dual modal operators $\Diamond_i$ are expressed as follows: $\Diamond_i = \neg\Box_i\neg$.

Definition 2.1 A *normal modal logic* (or *a logic,* for short) is a set of modal formulae closed under Substitution $\left(\frac{A(p_i)}{A(B)}\right)$, Modus Ponens $\left(\frac{A,\, A\to B}{B}\right)$ and Generalization rules $\left(\frac{A}{\Box_i A}\right)$; containing all classic tautologies and the following axioms:

$$\Box_i(p \to q) \to (\Box_i p \to \Box_i q).$$

$\mathsf{K_n}$ denotes *the minimal normal modal logic with n modalities* and $\mathsf{K} = \mathsf{K_1}$.

Let L be a logic and let Γ be a set of formulae, then $\mathsf{L} + \Gamma$ denotes the minimal logic containing L and Γ. If $\Gamma = \{A\}$, then we write $\mathsf{L} + A$ rather than $\mathsf{L} + \{A\}$.

Definition 2.2 Formula ϕ is called *closed* if it does not contain any variables.

Definition 2.3 Let $\mathsf{L_1}$ and $\mathsf{L_2}$ be two modal logics with one modality $\Box$, then *fusion* of these logics is

$$\mathsf{L_1} * \mathsf{L_2} = \mathsf{K_2} + \mathsf{L}'_1 + \mathsf{L}'_2;$$

where L_i' is the set of all formulae from L_i where all $\Box$ are replaced by $\Box_i$.

3 Products of neighborhood frames

The notions of Kripke frames and Kripke models are well known (see [2]).

Definition 3.1 Let $R \subseteq W \times W$ be a relation on $W \neq \varnothing$, then for $k \geq 1$ and $w \in W$ we define

$$\begin{aligned} R^0 &= Id_W; \\ R^{n+1} &= R^n \circ R; \\ R^* &= \bigcup_{k=0}^{\infty} R^k; \\ R(w) &= \{u \mid wRu\}. \end{aligned}$$

For a Kripke frame $F = (W, R)$ we define the submodel generated by $w \in W$ as the frame $F^w = (W', R|_{W'})$, where $W' = R^*(w)$ and $R|_{W'} = R \cap W' \times W'$.

Definition 3.2 Let $F_i = (W_i, R_i)$ $(i = 1, 2)$ be two Kripke frames. We define their *product* (see [4]) as a bimodal frame $F_1 \times F_2 = (W_1 \times W_2, R_1^h, R_2^v)$, where

$$\begin{aligned} (x, y)R_1^h(z, t) &\iff xR_1z \ \&\ y = t, \\ (x, y)R_2^v(z, t) &\iff yR_2t \ \&\ x = z. \end{aligned}$$

Furthermore, we consider neighborhood frames (see [10] and [3]).

Definition 3.3 Let X be a nonempty set, then $F \subseteq 2^X$ is a *filter* on X if

(i) $X \in F$;

(ii) if $U, V \in F$, then $U \cap V \in F$;

(iii) if $U \in F$ and $U \subseteq V$, then $V \in F$.

Note, it is usually demanded that $\varnothing \notin F$ (F is a proper filter), but in this paper we will not demand this.

Definition 3.4 A *(normal) neighborhood frame* (or an n-frame) is a pair $\mathfrak{X} = (X, \tau)$, where X is a nonempty set and $\tau : X \to 2^{2^X}$ such that $\tau(x)$ is a filter on X for any x. We call function τ the *neighborhood function* of $\mathfrak{X}$ and sets from $\tau(x)$ we call *neighborhoods of* x. *The neighborhood model* (n-model) is a pair $(\mathfrak{X}, \theta)$, where $\mathfrak{X} = (X, \tau)$ is an n-frame and $\theta : PROP \to 2^X$ is a *valuation.* In a similar way, we define *neighborhood 2-frame* (n-2-frame) as (X, τ_1, τ_2) such that $\tau_i(x)$ is a filter on X for any x, and a *n-2-model.*

Definition 3.5 *The valuation of a formula* φ at a point of an n-model $M = (\mathfrak{X}, \theta)$ is defined by induction. For Boolean connectives the definition is usual, so we omit it. For modalities the definition is as follows:

$$M, x \models \Box_i\psi \iff \exists\theta \in \tau_i(x)\forall y \in \theta(M, y \models \psi).$$

Formula is valid in an n-model M if it is valid at all points of M (notation $M \models \varphi$). Formula is valid in an n-frame $\mathfrak{X}$ if it is valid in all models based on $\mathfrak{X}$ (notation $\mathfrak{X} \models \varphi$). We write $\mathfrak{X} \models L$ if for any $\varphi \in L$, $\mathfrak{X} \models \varphi$. Logic of a class of n-frames $\mathcal{C}$ as $Log(\mathcal{C}) = \{\varphi \,|\, \mathfrak{X} \models \varphi \text{ for all } \mathfrak{X} \in \mathcal{C}\}$. For logic L we also define $nV(L) = \{\mathfrak{X} \,|\, \mathfrak{X} \text{ is an n-frame and } \mathfrak{X} \models L\}$. Note, that if there is no $\mathfrak{X}$ such that $\mathfrak{X} \models L$, then $nV(L) = \varnothing$.

Definition 3.6 Let $F = (W, R)$ be a Kripke frame. We define n-frame $\mathcal{N}(F) = (W, \tau)$ in the following way

$$\tau(w) = \{U \,|\, R(w) \subseteq U \subseteq W\}\,.$$

Lemma 3.7 *Let $F = (W, R)$ be a Kripke frame. Then*

$$Log(\mathcal{N}(F)) = Log(F).$$

The proof is straightforward (see [3]).

Definition 3.8 Let $\mathfrak{X} = (X, \tau_1, \ldots)$ and $\mathcal{Y} = (Y, \sigma_1, \ldots)$ be n-frames. Then function $f : X \to Y$ is a *p-morphism* if

(i) f is surjective;

(ii) for any $x \in X$ and $U \in \tau_i(x)$ $f(U) \in \sigma_i(f(x))$;

(iii) for any $x \in X$ and $V \in \sigma_i(f(x))$ there exists $U \in \tau_i(x)$ such that $f(U) \subseteq V$.

In notation $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$.

Remark 3.9 According to Lemma 3.7, a Kripke frame is a particular case of a neighborhood frame. There is a notion of p-morphism for Kripke frames. It is easy to check that for any two Kripke frames F and G function f is a p-morphism from F to G iff f is a p-morphism from $\mathcal{N}(F)$ to $\mathcal{N}(G)$. So, p-morphism for n-frames is a natural generalization of p-morphism for Kripke frames.

Lemma 3.10 *Let $\mathfrak{X} = (X, \tau_1, \ldots)$, $\mathcal{Y} = (Y, \sigma_1, \ldots)$ be n-frames and $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$. Let θ' be a valuation on $\mathcal{Y}$. We define $\theta(p) = f^{-1}(\theta'(p))$. Then*

$$\mathfrak{X}, \theta, x \models \varphi \iff \mathcal{Y}, \theta', f(x) \models \varphi.$$

The proof is by standard induction on the length of formula φ.

Corollary 3.11 *If $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$, then $Log(\mathfrak{X}) \subseteq Log(\mathcal{Y})$.*

Definition 3.12 Let $\mathfrak{X}_1 = (X_1, \tau_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2)$ be two n-frames. Then *the product* of these n-frames is an n-2-frame defined as follows:

$$\begin{array}{l}\mathfrak{X}_1 \times \mathfrak{X}_2 = (X_1 \times X_2, \tau_1', \tau_2'),\\ \tau_1'(x_1, x_2) = \{U \subseteq X_1 \times X_2 \,|\, \exists V (V \in \tau_1(x_1) \;\&\; V \times \{x_2\} \subseteq U)\}\,,\\ \tau_2'(x_1, x_2) = \{U \subseteq X_1 \times X_2 \,|\, \exists V (V \in \tau_2(x_2) \;\&\; \{x_1\} \times V \subseteq U)\}\,.\end{array}$$

Note that for normal n-frames $\mathfrak{X}_1$ and $\mathfrak{X}_2$ their product $\mathfrak{X}_1 \times \mathfrak{X}_2$ is also normal.

Definition 3.13 For two unimodal logics L_1 and L_2, so that $nV(L_1) \neq \varnothing$ and $nV(L_2) \neq \varnothing$, we define *n-product* of them as follows:

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = Log(\{\mathfrak{X}_1 \times \mathfrak{X}_2 \,|\, \mathfrak{X}_1 \in nV(L_1) \;\&\; \mathfrak{X}_2 \in nV(L_2)\}).$$

If we forget about one of its neighborhood functions, say τ_2', then $\mathfrak{X}_1 \times \mathfrak{X}_2$ will be a disjoint union of L_1 n-frames. Hence,

Proposition 3.14 ([8]) *For two unimodal normal logics* L_1 *and* L_2

$$\mathsf{L}_1 * \mathsf{L}_2 \subseteq \mathsf{L}_1 \times_n \mathsf{L}_2.$$

From [5] we know that n-product of any two logics from set $\{\mathsf{S4}, \mathsf{D4}, \mathsf{D}, \mathsf{T}\}$ equals to the fusion of corresponding logics. But this is not the case for K.

Proposition 3.15 $\mathsf{K} \times_n \mathsf{K} \neq \mathsf{K} * \mathsf{K}$.

Proof. Let $\mathfrak{X}_1 = (X_1, \tau_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2)$ be two n-frames and $\mathfrak{X}_1 \times \mathfrak{X}_2 = (X_1 \times X_2, \tau_1', \tau_2')$. Consider formula $\Box_1\bot \to \Box_2\Box_1\bot$. Since this formula has no variables the truth of this formula does not depend on the valuation. So

$$\begin{aligned} \mathfrak{X}_1 \times \mathfrak{X}_2, (x,y) \models \Box_1\bot \iff \varnothing \in \tau_1'(x,y) \iff \\ \varnothing \in \tau_1(x) \iff \forall y' \in X_2\ (\varnothing \in \tau_1'(x,y')) \iff \\ \forall y' \in X_2\ (\mathfrak{X}_1 \times \mathfrak{X}_2, (x,y') \models \Box_1\bot) \implies \mathfrak{X}_1 \times \mathfrak{X}_2, (x,y) \models \Box_2\Box_1\bot. \end{aligned}$$

Hence, $\mathfrak{X}_1 \times \mathfrak{X}_2 \models \Box_1\bot \to \Box_2\Box_1\bot$. □

Moreover,

Lemma 3.16 *For any two n-frames* $\mathfrak{X}_1$ *and* $\mathfrak{X}_2$ *1) if* ϕ *is a closed formula without* $\Box_2$*, then for any two n-frames* $\mathfrak{X}_1$ *and* $\mathfrak{X}_2$

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models \phi \to \Box_2\phi,$$

2) if ϕ *is a closed formula without* $\Box_1$*, then*

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models \phi \to \Box_1\phi.$$

Proof. We prove only 1) because 2) can be proved analogously. Since ϕ does not contain neither $\Box_2$, nor variables, its value does not depend on the second coordinate. Let $F = \mathfrak{X}_1 \times \mathfrak{X}_2$. So if $F, (x,y) \models \phi$, then $\forall y'(F, (x,y') \models \phi)$, hence, $F, (x,y) \models \Box_2\phi$. □

We put

$$\Delta = \{\phi \to \Box_2\phi \,|\, \phi \text{ is closed and } \Box_2\text{-free}\} \cup \{\psi \to \Box_1\psi \,|\, \psi \text{ is closed and } \Box_1\text{-free}\}.$$

Definition 3.17 For two unimodal logics L_1 and L_2, we define

$$\langle L_1, L_2 \rangle = L_1 * L_2 + \Delta.$$

From Lemma 3.16 and Proposition 3.14 follows:

Proposition 3.18 *For any two normal modal logics L_1 and L_2 $\langle L_1, L_2\rangle \subseteq L_1 \times_n L_2$.*

Corollary 3.19 $\langle \mathsf{K}, \mathsf{K}\rangle \subseteq \mathsf{K} \times_n \mathsf{K}$.

The rest of the paper is dedicated to proving the converse inclusion.

4 Weak product of Kripke frames

In order to prove completeness of $\langle \mathsf{K}, \mathsf{K}\rangle$ w.r.t. n-frames, we first establish completeness w.r.t. special kind of Kripke frames. For this purpose we use "weak product" of Kripke frames which basically is the result of unraveling (c.f. [2]) of the usual product of two Kripke frames.

Definition 4.1 Let $G = (W, R_1, R_2) = G^{w_0}$ be a 2-modal Kripke frame with root w_0. A *path* in G is a tuple $\delta = w_0 R_{i_1} w_1 \dots R_{i_k} w_k$, so that for any $j > 0$ $w_{j-1} R_{i_j} w_j$. Path δ is called an (n, m)-path if set $\{j \mid i_j = 1\}$ has no more than n elements and set $\{j \mid i_j = 2\}$ has no more than m elements.

Definition 4.2 Let F_1 and F_2 be two Kripke frames with roots x_0 and y_0 respectively. A *path* in the product $F_1 \times F_2$ is a sequence of the following type

$$(x_0, y_0) S_1 (x_1, y_1) S_2 \dots S_n (x_n, y_n),$$

where $S_i \in \{R_1^h, R_2^v\}$ and for any $i \le n$ $(x_{i-1}, y_{i-1}) S_i (x_i, y_i)$ holds.

Let $\mathcal{P}(F_1 \times F_2)$ be the set of all paths in $F_1 \times F_2$.

We define relations on $\mathcal{P}(F_1 \times F_2)$ in the following way: for any two paths α and β

$$\alpha R_1' \beta \iff \beta = \alpha R_1^h (a, b)$$
$$\alpha R_2' \beta \iff \beta = \alpha R_2^v (a, b)$$

We will call the following Kripke frame *weak product* of F_1 and F_2

$$\langle F_1, F_2\rangle = (\mathcal{P}(F_1 \times F_2), R_1', R_2').$$

Lemma 4.3 *For any two Kripke frames F_1 and F_2 $\langle F_1, F_2\rangle \models \Delta$.*

Proof. Let $\phi \to \Box_2 \phi \in \Delta$, i.e. ϕ is closed, $\Box_2$-free and $\alpha \models \phi$. Since ϕ is variable-free and $\Box_2$-free, its truth in $\langle F_1, F_2\rangle$ depends only on the structure of frame $G_\alpha^1 = (W_1 \times W_2, R_1')^\alpha = (R_1'^*(\alpha), R_1'|_{R_1'^*(\alpha)})$.

Due to the construction of $\langle F_1, F_2\rangle$ for any β, so that $\alpha R_2' \beta$, G_β^1 is isomorphic to G_α^1, so $\beta \models \phi$ and, hence, $\alpha \models \Box_2 \phi$.

Similarly, we prove that $\langle F_1, F_2\rangle \models \psi \to \Box_1 \psi$ for any closed $\Box_1$-free formula ψ. □

The aim of this section is to prove the following theorem:

Theorem 4.4 *Logic $\langle \mathsf{K}, \mathsf{K}\rangle$ is complete with respect to the class of all weak products of Kripke frames.*

In order to prove this theorem, we introduce some notions and constructions.

From here on in this section we rewrite all formulae using only $\Diamond$ instead of $\Box$.

Definition 4.5 For a modal formula ψ, we define its modal depth $d(\psi)$ as follows:

$$d(\bot) = d(p) = 0;$$
$$d(\psi_1 \to \psi_2) = \max(d(\psi_1), d(\psi_2));$$
$$d(\Diamond\psi) = d(\psi) + 1.$$

Let Σ be a consistent set of closed 1-modal formulae maximal up to depth n. We define frame $\mathcal{F}(\Sigma, n)$ of depth n by induction:

Base:

$$\mathcal{F}(\Sigma, 0) = (\{\bullet\}, \varnothing) \text{ — an irreflexive one-point frame.}$$

Step: assume that $\mathcal{F}(\Omega, n)$ is defined for any maximal up to depth n set of closed formulae Ω; and Σ is maximal up to depth $n+1$ set of closed formulae.

Let $\mathcal{O}_n$ be the set of all maximal consistent sets of closed formulae of depth not greater than n. Note that there are only finitely many nonequivalent closed formulae of depth not greater than n. Let $\Omega \in \mathcal{O}_n$, since Ω is finite, then we can define

$$\zeta_\Omega = \Diamond(\bigwedge \Omega).$$

Note that $d(\zeta_\Omega) \leq n+1$ and due to maximality of Σ either $\zeta_\Omega \in \Sigma$ or $\neg\zeta_\Omega \in \Sigma$.

Let $F^0_\Omega = \mathcal{F}(\Omega, n)$ and F^i_Ω be a copy of F^0_Ω for each $i \in \mathbb{N}$.

Definition 4.6 Let $G_0 = (W_0, R_0), G_1 = (W_1, R_1), \ldots$ be a finite or infinite set of Kripke frames, so that all W_i are disjoint. Then by $H = (\bullet) + \bigsqcup G_i$ we define Kripke frame $H = (V, S)$ in the following way:

$$V = \{r\} \cup \bigcup W_i,$$
$$S = \Big(\{r\} \times \bigcup W_i\Big) \cup \bigcup R_i.$$

Now we can define $\mathcal{F}(\Sigma, n+1)$

$$\mathcal{F}(\Sigma, n+1) = (\bullet) + \bigsqcup_{\zeta_\Omega \in \Sigma} \bigsqcup_{i \in \mathbb{N}} F^i_\Omega.$$

Definition 4.7 A Kripke frame $F = (W, R)$ is called *tree* with root r, if for any point $w \in W$, $w \neq r$, there is only one immediate predecessor and r has no predecessors.

Using standard unraveling method (see [2]) one can prove

Lemma 4.8 *For any countable Kripke frame F there exists a countable tree G such that $G \twoheadrightarrow F$.*

Let G be a tree of depth not greater than n. For $w \in G$ we define

$$\Sigma_n(w) = \{\psi \mid \psi \text{ is closed, } d(\psi) \leq n \text{ and } G, w \models \psi\}.$$

Note, that here we can write $G, w \models \psi$ (without valuation) because ψ is closed and does not depend on the valuation.

Lemma 4.9 *Let G be a countable tree, then for any $w \in G$ there exists $f : \mathcal{F}(\Sigma_n(w), n) \twoheadrightarrow G^w\lceil n$. Where $G^w\lceil n$ is the subframe of G^w, so that all points of depth greater than n are eliminated.*

Proof.

We construct f by induction.

Let I be the set of all successors of w. We split I into classes $I = \bigcup I_j$, so that for any j and any $u, u' \in I_j$ $\Sigma_{n-1}(u) = \Sigma_{n-1}(u')$. For each I_j. For each j we fix a surjective map $h_j : \mathbb{N} \to I_j$.

Remember, that

$$\mathcal{F}(\Sigma_n(w), n) = (\bullet) + \bigsqcup_{\zeta_\Omega \in \Sigma_n(w)} \bigsqcup_{i \in \mathbb{N}} F^i_\Omega, \text{ where } (\bullet) = (\{r\}, \varnothing). \tag{1}$$

We put

$$f(r) = w. \tag{2}$$

By induction for each j and $i \in \mathbb{N}$ there exists

$$g_{j,i} : F^i_{\Sigma_{n-1}(h_j(i))} \twoheadrightarrow G^{h_j(i)}\lceil(n-1).$$

For each Ω, so that $\zeta_\Omega \in \Sigma_n(w)$, there exist $u \in I_j$ such that $\Omega = \Sigma_{n-1}(u)$, and there exists i such that $h_j(i) = u$. So for any $x \in F^i_\Omega$ we put

$$f(x) = g_{j,i}(x). \tag{3}$$

Thus, (2) and (3) define f completely. Now we need to show that f is indeed a p-morphism. Let (V, S) be the frame from (1).

(i) Surjectiveness of f is obvious.

(ii) Let xSy and $x \neq r$, then $f(x)Rf(y)$ because for corresponding i and j $g_{j,i}$ is a p-morphism.

Now assume that $x = rSy$, then $y \in I_j$ for some j and $f(y)$ is a successor of w.

(iii) For $x \neq r$ it follows from the fact that all $g_{j,i}$ are p-morphisms.

Assume that $x = r$ and $f(r) = wRu$, then $u \in I_j$ for some j, and there exist an i such that $h_j(i) = u$. Hence, for the root r' of frame $F^i_{\Sigma_{n-1}(h_j(i))}$ $f(r') = u$ and rSr'.

□

Lemma 4.10 *If $\phi \notin \mathsf{K}$, then there is a set of closed formulae Σ such that $\mathcal{F}(\Sigma, d(\phi)) \not\models \phi$.*

Proof. It is well-known that logic K has countable (even finite) model property (see [2]). By Lemma 4.8 there is a countable tree $G = (W, R)$ with root w_0, so that $G, \theta, w_0 \models \neg\phi$ for some valuation θ.

Since the truth of ϕ depends only on points of depth not greater than $n = d(\phi)$ then $G\lceil n, \theta|_{G\lceil n}, w_0 \models \neg\phi$. Let $\Sigma = \Sigma_n(w_0)$. Σ is obviously a maximal consistent set of closed formulae up to depth n. By Lemma 4.9 $\mathcal{F}(\Sigma, n) \twoheadrightarrow G\lceil n$. Hence, $\mathcal{F}(\Sigma, n) \not\models \phi$. □

Corollary 4.11 *Logic* K *is complete with respect to the following class of frames:* $\{\mathcal{F}(\Sigma, n) \,|\, \Sigma \in \mathcal{O}_n,\ n \in \mathbb{N}\}$.

Let us go back to proving Theorem 4.4. We define $\Diamond_1$-depth $d_{\Diamond_1}$ and $\Diamond_2$-depth $d_{\Diamond_2}$ for any 2-modal formula:

$$
\begin{array}{rr}
d_{\Diamond_i}(\bot) = d_{\Diamond_i}(p) = 0; & d_{\Diamond_i}(\psi_1 \to \psi_2) = \max(d_{\Diamond_i}(\psi_1), d_{\Diamond_i}(\psi_2)); \\
d_{\Diamond_1}(\Diamond_1\psi) = d_{\Diamond_1}(\psi) + 1; & d_{\Diamond_2}(\Diamond_1\psi) = d_{\Diamond_2}(\psi); \\
d_{\Diamond_1}(\Diamond_2\psi) = d_{\Diamond_1}(\psi); & d_{\Diamond_2}(\Diamond_2\psi) = d_{\Diamond_2}(\psi) + 1;
\end{array}
$$

for any $i \in \{1, 2\}$.

Since the standard translation of a closed formula produces a first-order condition on frames, $\langle \mathsf{K}, \mathsf{K} \rangle$ is Δ-elementary. Therefore, by [4, Prop. 5.4], $\langle \mathsf{K}, \mathsf{K} \rangle$ is complete with respect to its countable rooted Kripke frames.

Assume that $\phi \notin \langle \mathsf{K}, \mathsf{K} \rangle$, then for a countable rooted Kripke frame F with root r and valuation θ, $F, \theta, r \models \neg\phi$. By Lemma 4.8 there is a 2-modal tree G such that $G \not\models \phi$.

For a 2-modal tree $G = (W, R_1, R_2)$ with root w_0 and $w \in W$, we define

$$
\begin{aligned}
\Sigma^1_n(w) &= \{\psi \,|\, \psi \text{ is closed and } \Box_2\text{-free, } d(\psi) \le n \text{ and } w \models \psi\}\,; \\
\Sigma^2_n(w) &= \{\psi \,|\, \psi \text{ is closed and } \Box_1\text{-free, } d(\psi) \le n \text{ and } w \models \psi\}\,.
\end{aligned}
$$

Let $F_1 = \mathcal{F}(\Sigma^1_n(w), n) = (V_1, S_1)$, $F_2 = \mathcal{F}(\Sigma^2_m(w), m) = (V_2, S_2)$, where $n = d_{\Diamond_1}(\phi)$ and $m = d_{\Diamond_2}(\phi)$. Then

Definition 4.12 Tree $G = (W, R_1, R_2)$ is called an (n, m)*-tree with root* w_0 if any point in W can be accessed from w_0 with an (n, m)-path.

Lemma 4.13 *Let G be an (n, m)-tree with root w_0, then there exist two uni-modal frames F_1, F_2 and a p-morphism $f : \langle F_1, F_2 \rangle \twoheadrightarrow G$.*

Proof. We will use induction on $n + m$. Let r_1 be the root of F_1 and r_2 be the root of F_2. We define

$$f(r_1, r_2) = w_0.$$

Let $H^1_w = (R^*_1(w), R_1|_{R^*_1(w)})$ and $H^2_w = (R^*_2(w), R_2|_{R^*_2(w)})$. By Lemma 4.9 there are $g_1 : F_1 \twoheadrightarrow H^1_{w_0}$ and $g_2 : F_2 \twoheadrightarrow H^2_{w_0}$.

Consider a path α in $F_1 \times F_2$ (an element of $\langle F_1, F_2 \rangle$). There are two possibilities:

1) $\alpha = (r_1, r_2)S'_1(u, r_2)\ldots = (r_1, r_2)S'_1\gamma$ and $r_1 S_1 u$, $g_1(u) = x$.

By induction, there is a p-morphism

$$h_u = \langle \mathcal{F}(\Sigma^1_{n-1}(x), n-1), \mathcal{F}(\Sigma^2_m(x), m) \rangle \twoheadrightarrow G^x.$$

Note that $\Sigma^1_{n-1}(x) = \Sigma_{n-1}(u)$. Let us show that $\Sigma^2_m(x) = \Sigma^2_m(w_0)$.

Indeed, if $\psi \in \Sigma^2_m(w_0)$, then $w_0 \models \psi$, but by Lemma 4.3 $w_0 \models \Box_1\psi$. Since $w_0 R_1 x$, then $x \models \psi$. So $\Sigma^2_m(w_0) \subseteq \Sigma^2_m(x)$ and, due to maximality, they are actually equal.

Therefore, cone of $\langle F_1, F_2 \rangle$ with root in (u, r_2) is isomorphic to $\langle \mathcal{F}(\Sigma^1_{n-1}(x), n-1), \mathcal{F}(\Sigma^2_m(x), m) \rangle$. Let t be this isomorphism.

We put

$$f(\alpha) = h_u(t(\alpha)).$$

2) $\alpha = (r_1, r_2)S'_2(r_1, v) \ldots = (r_1, r_2)S'_2\gamma$ and $r_2 S_2 v$, $g_2(v) = y$.

By induction, there is a p-morphism

$$h'_v = \langle \mathcal{F}(\Sigma^1_n(y), n), \mathcal{F}(\Sigma^2_{m-1}(y), m-1) \rangle \twoheadrightarrow G^y.$$

Similar to the previous case, there is an isomorphism

$$t' : \langle \mathcal{F}(\Sigma^1_n(y), n), \mathcal{F}(\Sigma^2_{m-1}(y), m-1) \rangle \to \langle F_1, F_2 \rangle^{(r_1, v)}.$$

So, we put

$$f(\alpha) = h'_v(t'(\alpha)).$$

Let us check that f is p-morphism also by induction:

Surjectiveness. Take any $y \in G_{(n,m)}$. If $y = w_0$, then its preimage is (r_1, r_2). Assume that $y \neq w_0$, then there is an (n, m)-path $\delta = w_0 R_k x \ldots y$ in $G_{(n,m)}$. Without loss of generality, we assume that $k = 1$ (case $k = 2$ is similar). By the construction, there exists u such that $f(u, r_2) = x$. Path $\eta = x \ldots y$ is an $(n-1, m)$-path, so that $\delta = w_0 R_k \eta$. h_u is surjective, hence there is a h_u-preimage of η and corresponding f-preimage of δ.

Monotonisity. Assume that δ and η are related in $\langle F_1, F_2 \rangle$ via the 1st relation, i.e. $\eta = \delta S'_1(v_1, v_2)$. If $\delta \neq (r_1, r_2)$, then monotonisity follows from monotonisity of h_u.

If $\delta = (r_1, r_2)$, then $\eta = (r_1, r_2)S'_1(v_1, r_2)$. By construction, $f(\delta) = w_0$, $f(\eta) = g_1(v_1)$ and $w_0 R_1 g_1(v_1)$.

For S'_2 the argument is the same.

Lifting. Assume that $f(\delta)R_1 y$. Since G is a tree, then there is only one predecessor of y, that is $f(\delta)$. If $f(\delta) \neq w_0$, then $f(\delta) = h_u(t(\delta))$ for some u. Since h_u is a p-morphism, then there exists γ such that $h_u(\gamma) = y$ and $t^{-1}(\gamma) = \delta S'_1(v_1, v_2)$.

If $f(\delta) = w_0$, then $g_1(u) = y$ for some u. So $\eta = (r_1, r_2)S'_1(u, r_2)$ satisfies the lifting condition.

□

To finish the proof of Theorem 4.4, note that $G_{(m,n)} \not\models \phi$ and by Lemma 4.13 there are F_1 and F_2 such that $\langle F_1, F_2 \rangle \not\models \phi$.

5 Completeness theorem

In this section we explain how, given two Kripke frames F_1 and F_2, to construct n-frames $\mathfrak{X}_1$ and $\mathfrak{X}_2$, so that $\mathfrak{X}_1 \times \mathfrak{X}_2 \twoheadrightarrow \mathcal{N}(\langle F_1, F_2\rangle)$. This is only possible if points in $\mathfrak{X}_1$ and $\mathfrak{X}_2$ do not have minimal neighborhoods or, in other words, each point should have arbitrary small neighborhoods. Because, otherwise, n-frames will be equivalent to Kripke frames, and we know that any product of Kripke frames satisfies commutativity axioms and Church-Rosser axiom. In order to construct such an n-frame, we introduce pseudo-infinite paths with stops.

Definition 5.1 For a frame $F = (W, R)$ with root a_0 we define *a path with stops* as a tuple $a_0 a_1 \dots a_n$, so that $a_i \in W$ or $a_i = 0$ and after eliminating zeros each point is related to the next one by relation R. We also consider infinite paths with stops that end with infinitely many zeros. We call these sequences *pseudo-infinite paths (with stops).* Let W_ω be the set of all pseudo-infinite paths in W.

Define $f_F : W_\omega \to W$ in the following way: for $\alpha = a_0 a_1 \dots a_n 0^\omega$, where 0^ω is an infinite sequence of zeros and $a_n \neq 0$, we put

$$f_F(\alpha) = a_n.$$

We also define

$$st(\alpha) = \min\{N \mid \forall k \geq N (a_k = 0)\};$$
$$\alpha|_k = a_1 \dots a_k;$$
$$U_i^k(\alpha) = \{\beta \in W_\omega \mid \alpha|_m = \beta|_m \ \&\ f_F(\alpha) R_i f_F(\beta), \text{ where } m = \max(k, st(\alpha))\}.$$

Lemma 5.2 *$U_i^k(\alpha) \subseteq U_i^m(\alpha)$ whenever $k \geq m$ for any $i \in \{1, 2\}$.*

Proof. Let $\beta \in U_i^k(\alpha)$. Since $\alpha|_k = \beta|_k$ and $k \geq m$, then $\alpha|_m = \beta|_m$. Hence, $\beta \in U_i^m(\alpha)$. □

Definition 5.3 Due to Lemma 5.2, sets $U_n(\alpha)$ form a filter base. So we can define

$$\tau(\alpha) - \text{the filter with base } \{U_n(\alpha) \mid n \in \mathbb{N}\};$$
$$\mathcal{N}_\omega(F) = (W_\omega, \tau) \text{ — is } \textit{a dense n-frame based on } F.$$

Frame $\mathcal{N}_\omega(F)$ is dense in a sense that the intersection of all neighborhoods of a point is empty. So, there are no minimal neighborhoods unlike $\mathcal{N}(F)$.

Lemma 5.4 *Let $F = (W, R)$ be a Kripke frame with root a_0, then*

$$f_F : \mathcal{N}_\omega(F) \twoheadrightarrow \mathcal{N}(F).$$

Proof. From now on in this proof we will omit the subindex in f_F. Since for any $b \in W$ there is a path $a_0 a_1 \dots b$ and, hence for pseudo-infinite path $\alpha = a_0 a_1 \dots b 0^\omega \in X$, $f(\alpha) = b$ and f is surjective.

Assume, that $\alpha \in W_\omega$ and $U \in \tau(\alpha)$. We need to prove that $R(f(\alpha)) \subseteq f(U)$. There exists m such that $U_m(\alpha) \subseteq U$ and since $f(U_m(\alpha)) = R(f(\alpha))$, then

$$R(f(\alpha)) = f(U_m(\alpha)) \subseteq f(U).$$

Assume that $\alpha \in W_\omega$ and V is a neighborhood of $f(\alpha)$, i.e. $R(f(\alpha)) \subseteq V$. We need to prove that there exists $U \in \tau(\alpha)$ such that $f(U) \subseteq V$. As U we take $U_m(\alpha)$ for some $m \geq st(\alpha)$, then

$$f(U_m(\alpha)) = R(f(\alpha)) \subseteq V.$$

□

Corollary 5.5 *For any frame F $Log(\mathcal{N}_\omega(F)) \subseteq Log(F)$.*

Proof. It follows from Lemmas 3.7, 5.4 and Corollary 3.11

$$Log(\mathcal{N}_\omega(F)) \subseteq Log(\mathcal{N}(F)) = Log(F).$$

□

Let $F_1 = (W_1, R_1) = F_1^{r_1}$ and $F_2 = (W_2, R_2) = F_2^{r_2}$ be two Kripke frames with roots. We assume that $W_1 \cap W_2 = \varnothing$. Consider the product of n-frames $\mathfrak{X}_1 = (X_1, \tau_1) = \mathcal{N}_\omega(F_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2) = \mathcal{N}_\omega(F_2)$

$$\mathfrak{X} = (X_1 \times X_2, \tau_1', \tau_2') = \mathcal{N}_\omega(F_1) \times_n \mathcal{N}_\omega(F_2).$$

We define function $g : \mathfrak{X}_1 \times \mathfrak{X}_2 \to \langle F_1, F_2 \rangle$ by induction, as follows.

Let $(\alpha, \beta) \in \mathfrak{X}_1 \times \mathfrak{X}_2$, so that $\alpha = x_1 x_2 \ldots$ and $\beta = y_1 y_2 \ldots$, $x_i \in W_1 \cup \{0\}$, $y_j \in W_2 \cup \{0\}$. We define $s(\alpha, \beta)$ to be the finite sequence that we get after eliminating all zeros from the infinite sequence $x_1 y_1 x_2 y_2 \ldots$. Now note, that we can uniquely map finite sequence $s(\alpha, \beta)$ to a path in $F_1 \times F_2$, because in $F_1 \times F_2$ we can only go up or right. Going up corresponds to adding a point from F_2, whereas going right corresponds to adding a point from F_1.

To be more precise, let $s(\alpha, \beta) = \mathbf{c} = w_1 w_2 \ldots w_n$. We define $h(\mathbf{c})$ by induction. If $\mathbf{c} = \varepsilon$ (empty string), then $h(\mathbf{c}) = (r_1, r_2)$. Assume that we already define $h(\mathbf{c}) = (x, y)$ and $\mathbf{b} = \mathbf{c}u$, then

$$h(\mathbf{b}) = \begin{cases} h(\mathbf{c}) R_1'(u, y) & \text{if } u \in W_1 \\ h(\mathbf{c}) R_2'(x, u) & \text{if } u \in W_2. \end{cases}$$

This definition is correct since, in the first case $x R_1 u$ and, in the second case, $y R_2 u$.

So we put $g(\alpha, \beta) = h(s(\alpha, \beta))$.

Lemma 5.6 *Function g defined above is a p-morphism: $g : \mathfrak{X} \twoheadrightarrow \mathcal{N}(\langle F_1, F_2 \rangle)$.*

Proof. Let $\mathbf{z} = (r_1, r_2) S_1 (z_1, t_1) S_2 \ldots S_n (z_n, t_n) \in \langle F_1, F_2 \rangle$. Define for $i \leq n$

$$x_i = \begin{cases} z_i, & \text{if } S_i = R_1'; \\ 0, & \text{if } S_i = R_2'; \end{cases} \qquad y_i = \begin{cases} 0, & \text{if } S_i = R_1'; \\ t_i, & \text{if } S_i = R_2'. \end{cases}$$

Let $\alpha = x_1x_2 \dots x_n0^\omega$ and $\beta = y_1y_2 \dots y_n0^\omega$, then $g(\alpha, \beta) = \mathbf{z}$. Hence g is surjective.

The next two conditions we check only for τ_1, since for τ_2 it is similar. Assume that $(\alpha, \beta) \in X_1 \times X_2$ and $U \in \tau_1(\alpha, \beta)$. We need to prove that $R_1'(g(\alpha, \beta)) \subseteq g(U)$. There exist $m > \max\{st(\alpha), st(\beta)\}$ such that $U_1^m(\alpha) \times \{\beta\} \subseteq U$ and, since $g(U_1^m(\alpha) \times \{\beta\}) = R_1'(g(\alpha, \beta))$, then

$$R_1'(g(\alpha, \beta)) = g(U_1^m(\alpha) \times \{\beta\}) \subseteq g(U);$$

where $U_1^m(\alpha)$ is the corresponding neighborhood from $\mathfrak{X}_1$.

Assume that $(\alpha, \beta) \in X_1 \times X_2$ and $R_1'(g(\alpha, \beta)) \subseteq V$. We need to prove that there exists $U \in \tau_1'(\alpha, \beta)$ such that $g(U) \subseteq V$. As U we take $U_m'(\alpha) \times \{\beta\}$ for some $m > \max\{st(\alpha), st(\beta)\}$, then

$$g(U_m'(\alpha) \times \{\beta\}) = R_1'(g(\alpha, \beta)) \subseteq V.$$

□

Corollary 5.7 *Let $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$, then $Log(\mathcal{N}_\omega(F_1) \times \mathcal{N}_\omega(F_2)) \subseteq Log(\langle F_1, F_2 \rangle)$.*

It immediately follows from Lemma 5.6 and Corollary 3.11.

Theorem 5.8 *Logic $\langle \mathsf{K}, \mathsf{K} \rangle$ is complete with respect to products of normal neighborhood frames, i.e.*

$$\langle \mathsf{K}, \mathsf{K} \rangle = \mathsf{K} \times_n \mathsf{K}. \tag{4}$$

Proof. The inclusion from left to rignt of (4) was proved in Corollary 3.19.

The converse inclusion follows from Theorem 4.4 and Corollary 5.7. Indeed

$$\begin{aligned} \mathsf{K} \times_n \mathsf{K} = & \bigcap_{\mathfrak{X}_1, \mathfrak{X}_2 \in nV(\mathsf{K})} Log(\mathfrak{X}_1 \times \mathfrak{X}_2) \subseteq \\ \subseteq & \bigcap_{F_1, F_2 - \text{Kripke frames}} Log(\mathcal{N}_\omega(F_1) \times \mathcal{N}_\omega(F_2)) \subseteq \\ \subseteq & \bigcap_{F_1, F_2 - \text{Kripke frames}} Log(\langle F_1, F_2 \rangle) \subseteq \langle \mathsf{K}, \mathsf{K} \rangle. \end{aligned}$$

□

6 Conclusion

Even though the logic $\langle \mathsf{K}, \mathsf{K} \rangle$ has infinite axiomatization, it is decidable. We will not go into details, but the argument is similar to the ones in [4]. To refute a formula ϕ we only need to consider frames of bounded depth, and standard argument shows that we can also assume bounded branching.

Even more, it seems that logic $\langle \mathsf{K}, \mathsf{K} \rangle$ has fmp in the class of weak products of Kripke frames. If the bounds turn out to be polinomial it, probably, will give us PSPACE completeness for $\langle \mathsf{K}, \mathsf{K} \rangle$.

The obvious next step is to apply these methods to other logics and try to prove completeness results for products of neighborhood frames for other logics. For example, we conjecture that $\langle \mathsf{K4}, \mathsf{K4} \rangle$ is the d-logic of all products of topological spaces.

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