# Completeness and Definability of a Modal Logic Interpreted over Iterated Strict Partial Orders


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**Abstract**

Any strict partial order $R$ on a nonempty set $X$ defines a function $\theta_R$ which associates to each strict partial order $S \subseteq R$ on $X$ the strict partial order $\theta_R(S) = R \circ S$ on $X$. Owing to the strong relationships between Alexandroff $T_D$ derivative operators and strict partial orders, this paper firstly calls forth the links between the Cantor-Bendixson ranks of Alexandroff $T_D$ topological spaces and the greatest fixpoints of the $\theta$-like functions defined by strict partial orders. It secondly considers a modal logic with modal operators $\square$ and $\square^\star$ respectively interpreted by strict partial orders and the greatest fixpoints of the $\theta$-like functions they define. It thirdly addresses the question of the complete axiomatization of this modal logic.



*Keywords:* Topologies; Derivative operators; Strict partial orders; Cantor-Bendixson rank; Modal logic; Completeness and definability.


## 1 Introduction

The $\tau$-derived set $d_\tau(A)$ of a set $A \subseteq X$ of points is the set of all limit points of $A$ with respect to a given topology $\tau$ on a nonempty set $X$. Introduced by Cantor, the derivative operator $d_\tau$ possesses interesting properties. In particular, a set $A \subseteq X$ of points is $\tau$-closed iff $d_\tau(A) \subseteq A$. A consequence of the entire description of $\tau$ in terms of derived sets is the possibility to use derivative operators $d$ as the primitive notion in topology. What happens if we iterate the derivative operator $d_\tau$, considering the sequence $d_\tau$, $d_\tau \circ d_\tau$, ... of operators? If $\tau$ is $T_D$ then each element $d_\tau^\alpha$ of this sequence is a derivative operator. Now,


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a question arises: what is the link between the topologies $\tau_\alpha$ corresponding to the elements $d_\tau^\alpha$ of the sequence? The answer is simple: the topologies $\tau_\alpha$ are getting finer when $\alpha$ increases. Since the lattice of all $T_D$ topologies on a given nonempty set $X$ is complete, this iteration process should stop. The Cantor-Bendixson rank of $(X, \tau)$ is then defined as the least ordinal $\alpha$ such that $d_\tau(d_\tau^\alpha(X)) = d_\tau^\alpha(X)$. A consequence of Tarski's fixpoint theorem [21] is that there exists an ordinal $\alpha^\star$ such that $\alpha \leq \alpha^\star$ and $d_\tau \circ d_\tau^{\alpha^\star} = d_\tau^{\alpha^\star}$, the greatest fixpoint of $d_\tau$.

Owing to the strong relationships between Alexandroff $T_D$ derivative operators and strict partial orders, the notion of rank of a strict partial order can also be defined. More precisely, any strict partial order $R$ on a given nonempty set $X$ defines a function $\theta_R$ which associates to each strict partial order $S \subseteq R$ on $X$ the strict partial order $\theta_R(S) = R \circ S$ on $X$. What happens if we iterate the function $\theta_R$, considering the sequence $R$, $\theta_R(R)$, ... of partial orders? Simply, the partial orders $\theta_R^\alpha(R)$ are getting smaller when $\alpha$ increases. Since the lattice of all strict partial orders on $X$ is complete, this iteration process should stop. And again, there exists an ordinal $\alpha^\star$ — called the rank of $R$ — such that $\theta_R(\theta_R^{\alpha^\star}(R)) = \theta_R^{\alpha^\star}(R)$, the greatest fixpoint of $\theta_R$. Moreover, if $R$ is the strict partial order on $X$ corresponding to a given Alexandroff $T_D$ derivative operator $d$, then $\theta_R^{\alpha^\star}(R)$ is a strict partial order on $X$ corresponding to the derivative operator $d_\tau^{\alpha^\star}$ considered above. Hence, it is natural to consider a modal logic with modal operators $\Box$ and $\Box^\star$ respectively interpreted by strict partial orders and the greatest fixpoints of the $\theta$-like functions they define. The goal of this paper is to address the question of its complete axiomatization.

Sections 2, 3 and 4 consider, on one hand, the strong relationships between topologies and derivative operators and, on the other hand, the strong relationships between Alexandroff $T_D$ derivative operators and strict partial orders. Most of the results they contain are well-known. See [8,9,11] for more on these. Sections 5, 6 and 7 present the above-mentioned modal logic and axiomatize it. The proof of its completeness is based on the step-by-step method.

## 2 Topologies and derivative operators

In this section, we present topologies and derivative operators. We also call forth the fact that topologies and derivative operators are the two sides of the same medal. See [8,9,11] for more on these.

### 2.1 Topologies

A *topology* on $X$ is a set $\tau$ of subsets of $X$ such that: (i) $\emptyset \in \tau$, (ii) $X \in \tau$, (iii) each union of members of $\tau$ is in $\tau$, (iv) each finite intersection of members of $\tau$ is in $\tau$. We shall say that $A \subseteq X$ is $\tau$-*closed* iff $X \setminus A \in \tau$. $\tau$ is said to be $T_D$ iff for all $x \in X$, there exists $A, B \in \tau$ such that $A \setminus B = \{x\}$. We shall say that $\tau$ is *Alexandroff* iff each intersection of members of $\tau$ is in $\tau$. Let $\leq$ be the binary relation between topologies on $X$ defined by $\tau \leq \tau'$ iff $\tau \subseteq \tau'$. It follows immediately from the definition that for all topologies $\tau, \tau'$ on $X$, if $\tau$ is $T_D$ and $\tau \leq \tau'$ then $\tau'$ is $T_D$.

**Example 2.1** If $X = \{x, y\}$ then let $\tau = \{\emptyset, \{x\}, X\}$, the Sierpiński space. Obviously, $\tau$ is a topology on $X$ such that the $\tau$-closed subsets of $X$ are $\emptyset$, $\{y\}$ and $X$. Moreover, since $\{x\} \setminus \emptyset = \{x\}$ and $X \setminus \{x\} = \{y\}$, $\tau$ is $T_D$. Finally, since $X$ is finite, $\tau$ is Alexandroff.

Given a topology $\tau$ on $X$, let $L_\tau$ be the set of all topologies $\tau'$ on $X$ such that $\tau \leq \tau'$. Remark that the least element of $L_\tau$ is $\tau$ and the greatest element of $L_\tau$ is the topology $\mathcal{P}(X)$. Moreover, the least upper bound of a family $\{\tau'_i\colon i \in I\}$ in $L_\tau$ is the intersection of all $\tau' \in L_\tau$ such that $\bigcup\{\tau'_i\colon i \in I\} \subseteq \tau'$ (note that the collection of all such $\tau'$ is nonempty, seeing that the topology $\mathcal{P}(X)$ belongs to it) and the greatest lower bound of a family $\{\tau'_i\colon i \in I\}$ in $L_\tau$ is $\bigcap\{\tau'_i\colon i \in I\}$. Hence, $(L_\tau, \leq)$ is a complete lattice.

### 2.2 Derivative operators

A *derivative operator* on $X$ is a function $d\colon \mathcal{P}(X) \to \mathcal{P}(X)$ such that: (i) $d(\emptyset) = \emptyset$, (ii) for all $A, B \subseteq X$, $d(A \cup B) = d(A) \cup d(B)$, (iii) for all $A \subseteq X$, $d(d(A)) \subseteq d(A) \cup A$, (iv) for all $x \in X$, $x \notin d(\{x\})$. $A \subseteq X$ is said to be $d$-closed iff $d(A) \subseteq A$. We shall say that $d$ is $T_D$ iff for all $A \subseteq X$, $d(d(A)) \subseteq d(A)$. $d$ is said to be *Alexandroff* iff for all $x \in X$, there exists a greatest $A \subseteq X$ such that $A$ is $d$-closed and $x \notin A$. Let $\leq$ be the binary relation between derivative operators on $X$ defined by $d \leq d'$ iff for all $A \subseteq X$, $d(A) \subseteq d'(A)$. It follows immediately from the definition and from the results stated in Section 2.3 that for all derivative operators $d, d'$ on $X$, if $d \leq d'$ and $d'$ is $T_D$ then $d$ is $T_D$.

**Example 2.2** If $X = \{x, y\}$ then let $d(\emptyset) = \emptyset$, $d(\{x\}) = \{y\}$, $d(\{y\}) = \emptyset$ and $d(X) = \{y\}$. Obviously, $d$ is a derivative operator on $X$ such that the $d$-closed subsets of $X$ are $\emptyset$, $\{y\}$ and $X$. Moreover, since $d(d(\emptyset)) \subseteq d(\emptyset)$, $d(d(\{x\})) \subseteq d(\{x\})$, $d(d(\{y\})) \subseteq d(\{y\})$ and $d(d(X)) \subseteq d(X)$, $d$ is $T_D$. Finally, since $X$ is finite, $d$ is Alexandroff.

Given a derivative operator $d$ on $X$, let $L_d$ be the set of all derivative operators $d'$ on $X$ such that $d' \leq d$. Remark that the least element of $L_d$ is the derivative operator $d_\emptyset\colon \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$, $d_\emptyset(A) = \emptyset$ and the greatest element of $L_d$ is $d$. What about the least upper bound of a family $\{d'_i\colon i \in I\}$ in $L_d$ and the greatest lower bound of a family $\{d'_i\colon i \in I\}$ in $L_d$? We do not know any representation of them using set-theoretic operations of the complete Boolean algebra of all subsets of $X$. Nevertheless, by the results stated in Sections 2.1 and 2.3, $(L_d, \leq)$ is a complete lattice.

### 2.3 Topologies v. derivative operators

Given a topology $\tau$ on $X$, let $d_\tau$ be the function $d_\tau\colon \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$, $d_\tau(A) = \{x\colon x \text{ is a } \tau\text{-limit point of } A\}$ where $x \in X$ is a $\tau$-*limit point* of $A \subseteq X$ iff for all $B \in \tau$, if $x \in B$ then $(B \setminus \{x\}) \cap A \neq \emptyset$. Remark that $d_\tau$ is a derivative operator on $X$ such that for all $A \subseteq X$, $A$ is $d_\tau$-closed iff $A$ is $\tau$-closed. Moreover, (i) $d_\tau$ is $T_D$ iff $\tau$ is $T_D$, (ii) $d_\tau$ is Alexandroff iff $\tau$ is Alexandroff, (iii) $d_{\tau'} \leq d_\tau$ iff $\tau \leq \tau'$.

**Example 2.3** If $X = \{x, y\}$ and $\tau$ is the topology on $X$ considered in Example 2.1 then $d_\tau$ is the derivative operator on $X$ considered in Example 2.2.

Given a derivative operator $d$ on $X$, let $\tau_d$ be the set of subsets of $X$ such that for all $A \subseteq X$, $A \in \tau_d$ iff $X \setminus A$ is $d$-closed. Remark that $\tau_d$ is a topology on $X$ such that for all $A \subseteq X$, $A$ is $\tau_d$-closed iff $A$ is $d$-closed. Moreover, (i) $\tau_d$ is $T_D$ iff $d$ is $T_D$, (ii) $\tau_d$ is Alexandroff iff $d$ is Alexandroff, (iii) $\tau_{d'} \leq \tau_d$ iff $d \leq d'$.

**Example 2.4** If $X = \{x, y\}$ and $d$ is the derivative operator on $X$ considered in Example 2.2 then $\tau_d$ is the topology on $X$ considered in Example 2.1.

To continue, let us further remark that $\tau_{d_\tau} = \tau$ and $d_{\tau_d} = d$. Given a topology $\tau$ on $X$, let $f$ be the function $f$: $L_\tau \to L_{d_\tau}$ such that $f(\tau') = d_{\tau'}$. By the results stated above, $f$ is an anti-isomorphism between $(L_{d_\tau}, \leq)$ and $(L_\tau, \leq)$. Given a derivative operator $d$ on $X$, let $f$ be the function $f$: $L_d \to L_{\tau_d}$ such that $f(d') = \tau_{d'}$. By the results stated above, $f$ is an anti-isomorphism between $(L_{\tau_d}, \leq)$ and $(L_d, \leq)$.

## 3 Alexandroff $T_D$ derivative operators and strict partial orders

In this section, we present Alexandroff $T_D$ derivative operators and strict partial orders. We also call forth the fact that Alexandroff $T_D$ derivative operators and strict partial orders are the two sides of the same medal. See [8,9,11] for more on these. In the sequel, if $R$ is a binary relation on a nonempty set $X$ then for all $x \in X$, $R(x)$ and $R^{-1}(x)$ will respectively denote the set of all $y \in X$ such that $xRy$ and the set of all $y \in X$ such that $yRx$. Moreover, for all $A \subseteq X$, $R(A)$ and $R^{-1}(A)$ will respectively denote the set $\bigcup\{R(x)\colon x \in A\}$ and the set $\bigcup\{R^{-1}(x)\colon x \in A\}$.

### 3.1 Alexandroff $T_D$ derivative operators

Given an Alexandroff $T_D$ derivative operator $d$ on $X$, let $L_d^A$ be the set of all Alexandroff $T_D$ derivative operators $d'$ on $X$ such that $d' \leq d$. Remark that the least element of $L_d^A$ is the derivative operator $d_\emptyset$ considered in Section 2.2 and the greatest element of $L_d^A$ is $d$. What about the least upper bound of a family $\{d_i'\colon i \in I\}$ in $L_d^A$ and the greatest lower bound of a family $\{d_i'\colon i \in I\}$ in $L_d^A$? We do not know any representation of them using set-theoretic operations of the complete Boolean algebra of all subsets of $X$. Nevertheless, by the results stated in Sections 3.2 and 3.3, $(L_d^A, \leq)$ is a complete lattice.

### 3.2 Strict partial orders

A *strict partial order* on $X$ is a binary relation $R$ on $X$ such that: (i) for all $x \in X$, $x \notin R(x)$, (ii) for all $x \in X$, $R(R(x)) \subseteq R(x)$. We shall say that $A \subseteq X$ is *R-closed* iff $R^{-1}(A) \subseteq A$. Let $\leq$ be the binary relation between strict partial orders on $X$ defined by $R \leq R'$ iff $R \subseteq R'$. Given a strict partial order $R$ on $X$, let $L_R$ be the set of all strict partial orders $R'$ on $X$ such that $R' \leq R$. Remark that the least element of $L_R$ is the strict partial order $\emptyset$ and the greatest element of $L_R$ is $R$. Moreover, the least upper bound of a family $\{R'_i\colon\ i \in I\}$ in $L_R$ is the transitive closure of $\bigcup\{R'_i\colon\ i \in I\}$ and the greatest lower bound of a family $\{R'_i\colon\ i \in I\}$ in $L_R$ is $\bigcap\{R'_i\colon\ i \in I\}$. Hence, $(L_R, \leq)$ is a complete lattice.

### 3.3 Alexandroff $T_D$ derivative operators v. strict partial orders

Given an Alexandroff $T_D$ derivative operator $d$ on $X$, let $R_d$ be the binary relation on $X$ such that for all $x, y \in X$, $xR_d y$ iff $x \in d(\{y\})$. Remark that $R_d$ is a strict partial order on $X$ such that for all $A \subseteq X$, $A$ is $R_d$-closed iff $A$ is $d$-closed. Moreover, $R_d \leq R_{d'}$ iff $d \leq d'$. Given a strict partial order $R$ on $X$, let $d_R$ be the function $d_R\colon\ \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$, $d_R(A) = R^{-1}(A)$. Remark that $d_R$ is an Alexandroff $T_D$ derivative operator on $X$ such that for all $A \subseteq X$, $A$ is $d_R$-closed iff $A$ is $R$-closed. Moreover, $d_R \leq d_{R'}$ iff $R \leq R'$. To continue, let us further remark that $d_{R_d} = d$ and $R_{d_R} = R$. Given an Alexandroff $T_D$ derivative operator $d$ on $X$, let $f$ be the function $f\colon\ L^A_d \to L_{R_d}$ such that $f(d') = R_{d'}$. By the results stated above, $f$ is an isomorphism between $(L_{R_d}, \leq)$ and $(L^A_d, \leq)$. Given a strict partial order $R$ on $X$, let $f\colon\ L_R \to L^A_{d_R}$ such that $f(R') = d_{R'}$. By the results stated above, $f$ is an isomorphism between $(L^A_{d_R}, \leq)$ and $(L_R, \leq)$.

## 4 Cantor-Bendixson ranks

In this section, we present Cantor-Bendixson ranks of Alexandroff $T_D$ derivative operators and strict partial orders.

### 4.1 Cantor-Bendixson ranks of Alexandroff $T_D$ derivative operators

Given an Alexandroff $T_D$ derivative operator $d$ on $X$, let $\theta_d$ be the function $\theta_d$: $L_d \to L_d$ such that for all $d' \in L_d$, $\theta_d(d') = d \circ d'$, i.e. $\theta_d(d')$ is the function $\theta_d(d')\colon\ \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$, $\theta_d(d')(A) = d(d'(A))$. Clearly, the function $\theta_d$ is monotonic. Since $(L_d, \leq)$ is a complete lattice, the function $\theta_d$ has a least fixpoint $\mathrm{lfp}(\theta_d)$ and a greatest fixpoint $\mathrm{gfp}(\theta_d)$. Obviously, $\mathrm{lfp}(\theta_d)$ is the derivative operator $d_\emptyset$ considered in Section 2.2. So, let us concentrate on $\mathrm{gfp}(\theta_d)$. A consequence of Tarski's fixpoint theorem [21] is that $\mathrm{gfp}(\theta_d)$ is the least upper bound of the family $\{d'\colon\ d' \leq \theta_d(d')\}$ in $L_d$. Next, we give the well-known characterization of $\mathrm{gfp}(\theta_d)$ in terms of ordinal powers of $\theta_d$. For all ordinals $\alpha$, we inductively define $\theta_d{\downarrow}\alpha$ as follows:

- $\theta_d{\downarrow}0$ is $d$,
- for all successor ordinals $\alpha$, $\theta_d{\downarrow}\alpha$ is $\theta_d(\theta_d{\downarrow}(\alpha - 1))$,
- for all limit ordinals $\alpha$, $\theta_d{\downarrow}\alpha$ is the greatest lower bound of the family $\{\theta_d{\downarrow}\beta\colon\ \beta \in \alpha\}$ in $L_d$.

The next result follows from the definition of $\theta_d{\downarrow}\alpha$ as being the greatest lower bound of the family $\{\theta_d{\downarrow}\beta\colon\ \beta \in \alpha\}$ in $L_d$ for each limit ordinal $\alpha$: (i) for all $x, y \in X$, $x \in \theta_d{\downarrow}\alpha(\{y\})$ iff for all ordinals $\beta$, if $\beta \in \alpha$ then $x \in \theta_d{\downarrow}\beta(\{y\})$, (ii) for all $A \subseteq X$, $\theta_d{\downarrow}\alpha(A) = \bigcap\{\theta_d{\downarrow}\beta(A)\colon\ \beta \in \alpha\}$. The next result is, again, a consequence of Tarski's fixpoint theorem [21]: (i) for all ordinals $\alpha$, $\mathrm{gfp}(\theta_d) \leq \theta_d{\downarrow}\alpha$, (ii) there exists an ordinal $\alpha$ such that $\mathrm{gfp}(\theta_d) = \theta_d{\downarrow}\alpha$. The least ordinal $\alpha$ such that $\theta_d{\downarrow}\alpha = \mathrm{gfp}(\theta_d)$ is called the *Cantor-Bendixson rank* of $d$.

**Example 4.1** If $X = \mathbb{Z}$ then let $d_\mathbb{Z}$ be the derivative operator on $X$ defined by $d_\mathbb{Z}(A) = \{x$: there exists $y \in X$ such that $x <_\mathbb{Z} y$ and $y \in A\}$ for each $A \subseteq X$. Obviously, $\theta_{d_\mathbb{Z}}(\theta_{d_\mathbb{Z}}{\downarrow}\omega) = \theta_{d_\mathbb{Z}}{\downarrow}\omega$. Moreover, no finite iteration of $\theta_{d_\mathbb{Z}}$ gives the greatest fixpoint. Hence, the Cantor-Bendixson rank of $d_\mathbb{Z}$ is $\omega$.

Remark that the Cantor-Bendixson rank of $d$ does not always coincide with the usual Cantor-Bendixson rank of the space $X$. Actually, it is the supremum of the Cantor-Bendixson ranks of all subspaces of $X$.

### 4.2 Cantor-Bendixson ranks of strict partial orders

Given a strict partial order $R$ on $X$, let $\theta_R$ be the function $\theta_R$: $L_R \rightarrow L_R$ such that for all $R' \in L_R$, $\theta_R(R') = R \circ R'$, i.e. $\theta_R(R')$ is the binary relation on $X$ such that for all $x, y \in X$, $x\theta_R(R')y$ iff there exists $z \in X$ such that $xRz$ and $zR'y$. Clearly, the function $\theta_R$ is monotonic. Since $(L_R, \leq)$ is a complete lattice, the function $\theta_R$ has a least fixpoint $\mathrm{lfp}(\theta_R)$ and a greatest fixpoint $\mathrm{gfp}(\theta_R)$. Obviously, $\mathrm{lfp}(\theta_R)$ is the strict partial order $\emptyset$. So, let us concentrate on $\mathrm{gfp}(\theta_R)$. A consequence of Tarski's fixpoint theorem [21] is that $\mathrm{gfp}(\theta_R)$ is the least upper bound of the family $\{R'$: $R' \leq \theta_R(R')\}$ in $L_R$. Next, we give the well-known characterization of $\mathrm{gfp}(\theta_R)$ in terms of ordinal powers of $\theta_R$. For all ordinals $\alpha$, we inductively define $\theta_R{\downarrow}\alpha$ as follows:

- $\theta_R{\downarrow}0$ is $R$,
- for all successor ordinals $\alpha$, $\theta_R{\downarrow}\alpha$ is $\theta_R(\theta_R{\downarrow}(\alpha - 1))$,
- for all limit ordinals $\alpha$, $\theta_R{\downarrow}\alpha$ is the greatest lower bound of the family $\{\theta_R{\downarrow}\beta$: $\beta \in \alpha\}$ in $L_R$.

The next result follows from the definition of $\theta_R{\downarrow}\alpha$ as being the greatest lower bound of the family $\{\theta_R{\downarrow}\beta$: $\beta \in \alpha\}$ in $L_R$ for each limit ordinal $\alpha$: (i) for all $x, y \in X$, $x\theta_R{\downarrow}\alpha y$ iff for all ordinals $\beta$, if $\beta \in \alpha$ then $x\theta_R{\downarrow}\beta y$, (ii) for all $A \subseteq X$, $\theta_R{\downarrow}\alpha^{-1}(A) \subseteq \bigcap\{\theta_R{\downarrow}\beta^{-1}(A)$: $\beta \in \alpha\}$. The next result is, again, a consequence of Tarski's fixpoint theorem [21]: (i) for all ordinals $\alpha$, $\mathrm{gfp}(\theta_R) \leq \theta_R{\downarrow}\alpha$, (ii) there exists an ordinal $\alpha$ such that $\mathrm{gfp}(\theta_R) = \theta_R{\downarrow}\alpha$. The least ordinal $\alpha$ such that $\theta_R{\downarrow}\alpha = \mathrm{gfp}(\theta_R)$ is called the *Cantor-Bendixson rank of R*.

**Example 4.2** If $X = \mathbb{Q}$ then let $R_\mathbb{Q}$ be the strict partial order on $X$ defined by $xR_\mathbb{Q}y$ iff $x <_\mathbb{Q} y$ for each $x, y \in X$. Obviously, $\theta_{R_\mathbb{Q}}(\theta_{R_\mathbb{Q}}{\downarrow}0) = \theta_{R_\mathbb{Q}}{\downarrow}0$. Hence, the Cantor-Bendixson rank of $R_\mathbb{Q}$ is 0.

### 4.3 Alexandroff $T_D$ derivative operators v. strict partial orders

Let $d$ be an Alexandroff $T_D$ derivative operator on $X$ and $R$ be a strict partial order on $X$ such that for all $x, y \in X$, $xRy$ iff $x \in d(\{y\})$ and for all $A \subseteq X$, $d(A) = R^{-1}(A)$. By the results stated in Sections 4.1 and 4.2, one can prove by induction on the ordinal $\alpha$ that (i) for all $x, y \in X$, $x\theta_R{\downarrow}\alpha y$ iff $x \in \theta_d{\downarrow}\alpha(\{y\})$,

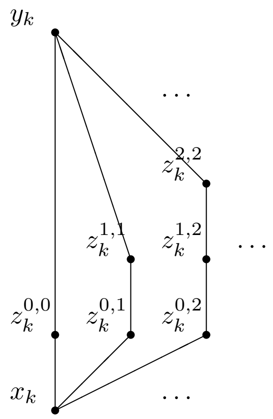


Fig. 1. The relational structure $(X_k, <_k)$.

(ii) for all $A \subseteq X$, $\theta_d{\downarrow}\alpha(A) \supseteq \theta_R{\downarrow}\alpha^{-1}(A)$. Let $\alpha_d$ be the Cantor-Bendixson rank of $d$ and $\alpha_R$ be the Cantor-Bendixson rank of $R$. The above considerations prove that (i) for all $x, y \in X$, $x\theta_R{\downarrow}\alpha_R y$ iff $x \in \theta_d{\downarrow}\alpha_d(\{y\})$, (ii) for all $A \subseteq X$, $\theta_d{\downarrow}\alpha_d(A) \supseteq \theta_R{\downarrow}{\alpha_R}^{-1}(A)$. Example 4.3 shows that the last inclusion can be strict.

**Example 4.3** For all $k \in \mathbb{N}$, let $X_k = \{x_k, y_k\} \cup \{z_k^{i,j}$: $i, j \in \mathbb{N}$ are such that $0 \leq i \leq j\}$ and $<_k$ be the least transitive relation on $X_k$ such that: (i) for all $i, j \in \mathbb{N}$ such that $0 \leq i \leq j$, $x_k <_k z_k^{i,j}$, (ii) for all $i_1, j_1, i_2, j_2 \in \mathbb{N}$ such that $0 \leq i_1 \leq j_1$ and $0 \leq i_2 \leq j_2$, $z_k^{i_1,j_1} <_k z_k^{i_2,j_2}$ iff $i_1 < i_2$ and $j_1 = j_2$, (iii) for all $i, j \in \mathbb{N}$ such that $0 \leq i \leq j$, $z_k^{i,j} <_k y_k$. See Figure 1. Take $X = \bigcup\{X_k$: $k \in \mathbb{N}\}$. Let $d$ be the function $d$: $\mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$, $d(A) = \{x$: there exists $y \in X$ such that $x < y$ and $y \in A\}$ and $R$ be the least transitive relation on $X$ such that: (i) for all $k \in \mathbb{N}$, $<_k \subseteq R$, (ii) for all $k, l \in \mathbb{N}$, if $k < l$ then $x_k R x_l$. Obviously, $d$ is a derivative operator on $X$ and $R$ is a strict partial order on $X$. Moreover, the Cantor-Bendixson ranks of $d$ and $R$ are both equal to $\omega + \omega$. Finally, $\theta_d{\downarrow}(\omega + \omega)$ is not the derivative operator $d_\emptyset$ considered in Section 2.2 and $\theta_R{\downarrow}(\omega + \omega)$ is the strict partial order $\emptyset$.

## 5 A modal logic

In this section, we present a modal logic with modal operators $\Box$ and $\Box^\star$. Section 5.2 presents the relational semantics where $\Box$ and $\Box^\star$ are respectively interpreted by strict partial orders and the greatest fixpoints of the $\theta$-like functions they define whereas Section 5.3 presents the topological semantics where $\Box$ and $\Box^\star$ are respectively interpreted by Alexandroff $T_D$ derivative operators and the greatest fixpoints of the $\theta$-like functions they define. Note that by 1944, McKinsey and Tarski [17] had already given an interpretation of $\Box$ in terms of derivative operators. For more on this, see also [3,11,19]. We assume the reader is at home with tools and techniques in modal logic; see [4,6,14] for more on these.

### 5.1 Syntax

The language is defined using a countable set $BV$ of Boolean variables (with typical members denoted by $p$, $q$, ...). We inductively define the set $f(BV)$ of *formulas* (with typical members denoted by $\phi$, $\psi$, ...) as follows:

- $\phi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi \mid \Box^\star\phi$.

The other Boolean constructs are defined as usual. We obtain the formulas $\Diamond\phi$ and $\Diamond^\star\phi$ as abbreviations: $\Diamond\phi ::= \neg\Box\neg\phi$, $\Diamond^\star\phi ::= \neg\Box^\star\neg\phi$. The notion of subformula is standard. We adopt the standard rules for omission of the parentheses.

### 5.2 Relational semantics

A *relational frame* is a structure of the form $\mathcal{F} = (X, R, S)$ such that (i) $X$ is a nonempty set, (ii) $R$ is a strict partial order on $X$, (iii) $S$ is the greatest fixpoint of the function $\theta_R$ in $L_R$. The following lemma is basic.

**Lemma 5.1** *Let $\mathcal{F} = (X, R, S)$ be a relational frame. (i) $R \circ R \leq R$, (ii) $S \circ S \leq S$, (iii) $S \leq R$, (iv) $R \circ S \leq S$, (v) $S \circ R \leq S$, (vi) $S \leq R \circ S$.*

**Proof.** (i), (ii) and (iii) follow from the fact that $R$ is a strict partial order on $X$, $S$ is a strict partial order on $X$ and $S \in L_R$. (iv), (v) and (vi) follow from the fact that $S$ is the greatest fixpoint of the function $\theta_R$ in $L_R$. $\Box$

A *relational model* is a structure of the form $\mathcal{M} = (X, R, S, V)$ where (i) $(X, R, S)$ is a relational frame, (ii) $V$ is a *valuation* on $X$, i.e. a function $V\colon BV \to \mathcal{P}(X)$. The *satisfiability* of $\phi \in f(BV)$ in a relational model $\mathcal{M} = (X, R, S, V)$ at $x \in X$, in symbols $\mathcal{M}, x \models \phi$, is inductively defined as follows:

- $\mathcal{M}, x \models p$ iff $x \in V(p)$,
- $\mathcal{M}, x \not\models \bot$,
- $\mathcal{M}, x \models \neg\phi$ iff $\mathcal{M}, x \not\models \phi$,
- $\mathcal{M}, x \models \phi \vee \psi$ iff either $\mathcal{M}, x \models \phi$ or $\mathcal{M}, x \models \psi$,
- $\mathcal{M}, x \models \Box\phi$ iff for all $y \in X$, if $xRy$ then $\mathcal{M}, y \models \phi$,
- $\mathcal{M}, x \models \Box^\star\phi$ iff for all $y \in X$, if $xSy$ then $\mathcal{M}, y \models \phi$.

As a result: $\mathcal{M}, x \models \Diamond\phi$ iff there exists $y \in X$ such that $xRy$ and $\mathcal{M}, y \models \phi$, $\mathcal{M}, x \models \Diamond^\star\phi$ iff there exists $y \in X$ such that $xSy$ and $\mathcal{M}, y \models \phi$. $\phi \in f(BV)$ is said to be *true* in a relational model $\mathcal{M} = (X, R, S, V)$, in symbols $\mathcal{M} \models \phi$, iff for all $x \in X$, $\mathcal{M}, x \models \phi$. We shall say that $\phi \in f(BV)$ is *valid* in a relational frame $\mathcal{F} = (X, R, S)$, in symbols $\mathcal{F} \models \phi$, iff for all valuations $V$ on $X$, $(X, R, S, V) \models \phi$. It is worth noting at this point the following:

**Lemma 5.2** *Let $\mathcal{F} = (X, R, S)$ be a relational frame. The following formulas are valid in $\mathcal{F}$: $\Box\phi \to \Box\Box\phi$, $\Box^\star\phi \to \Box^\star\Box^\star\phi$, $\Box\phi \to \Box^\star\phi$, $\Box^\star\phi \to \Box\Box^\star\phi$,*

$\Box^\star\phi \to \Box^\star\Box\phi$, $\Box\Box^\star\phi \to \Box^\star\phi$.

**Proof.** The above formulas are Sahlqvist formulas. By Sahlqvist Correspondence Theorem [4, Theorem 3.54], they correspond to the first-order conditions considered in Lemma 5.1. Hence, they are valid in $\mathcal{F}$. □

Let $\Lambda_{rf}$ be the set of all formulas that are valid in the class of all relational frames.

### 5.3 Topological semantics

A *topological frame* is a structure of the form $\mathcal{F} = (X, d, e)$ such that (i) $X$ is a nonempty set, (ii) $d$ is an Alexandroff $T_D$ derivative operator on $X$, (iii) $e$ is the greatest fixpoint of the function $\theta_d$ in $L_d$. The following lemma is basic.

**Lemma 5.3** *Let $\mathcal{F} = (X, d, e)$ be a topological frame. (i) $d \circ d \leq d$, (ii) $e \circ e \leq e$, (iii) $e \leq d$, (iv) $d \circ e \leq e$, (v) $e \circ d \leq e$, (vi) $e \leq d \circ e$.*

**Proof.** (i), (ii) and (iii) follow from the fact that $d$ is an Alexandroff $T_D$ derivative operator on $X$, $e$ is an Alexandroff $T_D$ derivative operator on $X$ and $e \in L_d$. (iv), (v) and (vi) follow from the fact that $e$ is the greatest fixpoint of the function $\theta_d$ in $L_d$. □

A *topological model* is a structure of the form $\mathcal{M} = (X, d, e, V)$ where (i) $(X, d, e)$ is a topological frame, (ii) $V$ is a *valuation* on $X$, i.e. a function $V\colon BV \to \mathcal{P}(X)$. The interpretation of $\phi \in f(BV)$ in a topological model $\mathcal{M} = (X, d, e, V)$, in symbols $\parallel \phi \parallel_{\mathcal{M}}$, is inductively defined as follows:

- $\parallel p \parallel_{\mathcal{M}} = V(p)$,
- $\parallel \bot \parallel_{\mathcal{M}} = \emptyset$,
- $\parallel \neg\phi \parallel_{\mathcal{M}} = X \setminus \parallel \phi \parallel_{\mathcal{M}}$,
- $\parallel \phi \vee \psi \parallel_{\mathcal{M}} = \parallel \phi \parallel_{\mathcal{M}} \cup \parallel \psi \parallel_{\mathcal{M}}$,
- $\parallel \Box\phi \parallel_{\mathcal{M}} = X \setminus d(X \setminus \parallel \phi \parallel_{\mathcal{M}})$,
- $\parallel \Box^\star\phi \parallel_{\mathcal{M}} = X \setminus e(X \setminus \parallel \phi \parallel_{\mathcal{M}})$.

As a result: $\parallel \Diamond\phi \parallel_{\mathcal{M}} = d(\parallel \phi \parallel_{\mathcal{M}})$, $\parallel \Diamond^\star\phi \parallel_{\mathcal{M}} = e(\parallel \phi \parallel_{\mathcal{M}})$. $\phi \in f(BV)$ is said to be *true* in a topological model $\mathcal{M} = (X, d, e, V)$, in symbols $\mathcal{M} \models \phi$, iff $\parallel \phi \parallel_{\mathcal{M}} = X$. We shall say that $\phi \in f(BV)$ is *valid* in a topological frame $\mathcal{F} = (X, d, e)$, in symbols $\mathcal{F} \models \phi$, iff for all valuations $V$ on $X$, $(X, d, e, V) \models \phi$. It is worth noting at this point the following:

**Lemma 5.4** *Let $\mathcal{F} = (X, d, e)$ be a topological frame. The following formulas are valid in $\mathcal{F}$: $\Box\phi \to \Box\Box\phi$, $\Box^\star\phi \to \Box^\star\Box^\star\phi$, $\Box\phi \to \Box^\star\phi$, $\Box^\star\phi \to \Box\Box^\star\phi$, $\Box^\star\phi \to \Box^\star\Box\phi$, $\Box\Box^\star\phi \to \Box^\star\phi$.*

**Proof.** The above formulas are Sahlqvist formulas. By Sahlqvist Correspondence Theorem [18], they correspond to the conditions considered in Lemma 5.3. Hence, they are valid in $\mathcal{F}$. □

Let $\Lambda_{tf}$ be the set of all formulas that are valid in the class of all topological frames.

# 6 Axiomatization and completeness

In this section, we present a complete axiomatization of $\Lambda_{rf}$.

## 6.1 Axiomatization

Let $L$ be the least normal modal logic in our language containing the formulas considered in Lemmas 5.2 and 5.4:

- $\Box\phi \rightarrow \Box\Box\phi$,
- $\Box^\star\phi \rightarrow \Box^\star\Box^\star\phi$,
- $\Box\phi \rightarrow \Box^\star\phi$,
- $\Box^\star\phi \rightarrow \Box\Box^\star\phi$,
- $\Box^\star\phi \rightarrow \Box^\star\Box\phi$,
- $\Box\Box^\star\phi \rightarrow \Box^\star\phi$.

Since these formulas are valid in the class of all relational frames and in the class of all topological frames,

**Proposition 6.1** *Let $\phi \in f(BV)$. If $\phi \in L$ then $\phi \in \Lambda_{rf}$ and $\phi \in \Lambda_{tf}$.*

It follows that $L$ is sound with respect to the class of all relational frames and with respect to the class of all topological frames. In spite of the connection between Alexandroff $T_D$ derivative operators and strict partial orders studied in Section 3, the class of all relational frames and the class of all topological frames do not validate the same formulas. By Proposition 6.1 and Theorem 6.17, $\Lambda_{rf} \subseteq \Lambda_{tf}$. Example 6.2 shows that the inclusion is strict (David Gabelaia, personal communication, Tbilisi (Georgia), March 24, 2012).

**Example 6.2** Let $\phi = \Box(p \rightarrow \Diamond p) \rightarrow (\Diamond p \rightarrow \Diamond^\star p)$, we demonstrate $\phi \notin \Lambda_{rf}$ and $\phi \in \Lambda_{tf}$. Intuitively, $\phi$ says that, in a relational frame $\mathcal{F} = (X, R, S)$, if we have an infinite sequence $y_0 R y_1 R \ldots$ then there exists $i, j \in \mathbb{N}$ such that $0 \leq i \leq j$ and $y_i S y_j$. Firstly, let $\mathcal{M} = (\mathbb{Z}, <_{\mathbb{Z}}, \emptyset, V)$ be the model defined over the integers and such that for all $q \in BV$, $V(q) = \mathbb{Z}$ and $x \in \mathbb{Z}$, we demonstrate $\mathcal{M}, x \not\models \phi$. Obviously, $\mathcal{M}, x \models \Box(p \rightarrow \Diamond p)$, $\mathcal{M}, x \models \Diamond p$ and $\mathcal{M}, x \not\models \Diamond^\star p$. Hence, $\mathcal{M}, x \not\models \phi$. Secondly, let $\mathcal{M} = (X, d, e, V)$ be a topological model, we demonstrate $\mathcal{M} \models \phi$. It suffices to demonstrate that $\parallel \phi \parallel_{\mathcal{M}} = X$, i.e. $d(V(p)) \setminus d(V(p) \setminus d(V(p))) \subseteq e(V(p))$. Let $A = d(V(p)) \setminus d(V(p) \setminus d(V(p)))$. Obviously, $A \subseteq d(V(p))$. Moreover, by [9, Section 8.5], $A \subseteq d(A)$. Thus, $d(A) \subseteq d(d(A))$. Since $d$ is a $T_D$ derivative operator on $X$, $d(d(A)) \subseteq d(A)$. Since $d(A) \subseteq d(d(A))$, $d(A) = d(d(A))$. Since $e$ is the greatest fixpoint of the function $\theta_d$ in $L_d$, $e(A) = d(A)$. Since $A \subseteq d(A)$, $A \subseteq e(A)$. Since $A \subseteq d(V(p))$, $e(A) \subseteq e(d(V(p)))$. Since $e$ is the greatest fixpoint of the function $\theta_d$ in $L_d$, $e(d(V(p)) \subseteq e(V(p))$. Since $e(A) \subseteq e(d(V(p)))$, $e(A) \subseteq e(V(p))$. Since $A \subseteq e(A)$, $A \subseteq e(V(p))$.

In the sequel, all frames and all models will be relational. The completeness of $L$ with respect to the class of all frames is more difficult to establish that its soundness and we defer proving it till the end of this section. $\Gamma \subseteq f(BV)$ is said to be an*L-theory* iff $\Gamma$ contains $L$ and $\Gamma$ is closed under the rule of modus ponens. Let us be clear that the set of all $L$-theories is a partially ordered set with respect to set inclusion. The least $L$-theory is $L$ and the greatest $L$-theory is $f(BV)$. Of course, an $L$-theory $\Gamma$ is equal to $f(BV)$ iff $\bot \in \Gamma$. We shall say that an $L$-theory $\Gamma$ is *consistent* iff $\bot \notin \Gamma$. $\phi \in f(BV)$ is said to be *L-consistent* iff there exists a consistent $L$-theory $\Gamma$ such that $\phi \in \Gamma$. Of course, $\phi \in f(BV)$ is $L$-consistent iff $\neg\phi \notin L$. We shall say that an $L$-theory $\Gamma$ is *maximal* iff for all $\phi \in f(BV)$, either $\phi \in \Gamma$, or $\neg\phi \in \Gamma$. The set of all maximal consistent $L$-theories will be denoted $MCT_L$. For all $L$-theories $\Gamma$ and for all $\phi \in f(BV)$, let $\Gamma + \phi = \{\psi\colon \phi \to \psi \in \Gamma\}$. For all $L$-theories $\Gamma$, let $\Box\Gamma = \{\phi\colon \Box\phi \in \Gamma\}$ and $\Box^\star\Gamma = \{\phi\colon \Box^\star\phi \in \Gamma\}$. One can easily establish the following results.

**Lemma 6.3** *Let $\Gamma$ be an $L$-theory. (i) For all $\phi \in f(BV)$, $\Gamma + \phi$ is the least $L$-theory containing $\Gamma$ and $\phi$, (ii) for all $\phi \in f(BV)$, $\Gamma + \phi$ is consistent iff $\neg\phi \notin \Gamma$, (iii) $\Box\Gamma$ is an $L$-theory, (iv) $\Box^\star\Gamma$ is an $L$-theory.*

Our next results are variants of Lindenbaum's Lemma [4, Lemma 4.17] and the Existence Lemma [4, Lemma 4.20].

**Lemma 6.4** *Let $\Gamma$ be a consistent $L$-theory. There exists $\Delta \in MCT_L$ such that $\Gamma \subseteq \Delta$.*

**Lemma 6.5** *Let $\Gamma \in MCT_L$ and $\phi \in f(BV)$. (i) If $\Box\phi \notin \Gamma$ then there exists $\Delta \in MCT_L$ such that $\Box\Gamma \subseteq \Delta$ and $\phi \notin \Delta$, (ii) if $\Box^\star\phi \notin \Gamma$ then there exists $\Delta \in MCT_L$ such that $\Box^\star\Gamma \subseteq \Delta$ and $\phi \notin \Delta$.*

Moreover,

**Lemma 6.6** *Let $\Gamma, \Delta \in MCT_L$. If $\Box^\star\Gamma \subseteq \Delta$ then there exists $\Lambda \in MCT_L$ such that $\Box\Gamma \subseteq \Lambda$ and $\Box^\star\Lambda \subseteq \Delta$.*

**Proof.** The proof is very similar to the one considered, for example, in [12, Theorem 3.6] to derive density conditions. □

What we have in mind is to demonstrate that if $\phi \in f(BV)$ is valid in the class of all frames then $\phi \in L$. In this respect, the concept of subordination structure will be needed. A *subordination structure* is a structure of the form $\mathcal{S} = (X, R, S, \mu)$ where (i) $X$ is a finite nonempty subset of $\mathbb{Z}$, (ii) $R$ is a strict partial order on $X$, (iii) $S$ is a strict partial order on $X$, (iv) $S \subseteq R$, (v) $R \circ S \subseteq S$, (vi) $S \circ R \subseteq S$, (vii) $\mu$ is an *interpretation* on $X$, i.e. a function $\mu\colon X \to MCT_L$ such that (vii-a) for all $x, y \in X$, if $xRy$ then $\Box\mu(x) \subseteq \mu(y)$, (vii-b) for all $x, y \in X$, if $xSy$ then $\Box^\star\mu(x) \subseteq \mu(y)$. $\phi \in f(BV)$ is said to be *true* in a subordination structure $\mathcal{S} = (X, R, S, \mu)$, in symbols $\mathcal{S} \models \phi$, iff for all $x \in X$, $\phi \in \mu(x)$. Given two subordination structures $\mathcal{S} = (X, R, S, \mu)$ and $\mathcal{S}' = (X', R', S', \mu')$, we shall say that $\mathcal{S}'$ *contains* $\mathcal{S}$, in symbols $\mathcal{S} \ll \mathcal{S}'$, iff $X \subseteq X'$, $R \subseteq R'$, $S \subseteq S'$ and for all $x \in X$, $\mu(x) = \mu'(x)$. In a subordination structure $\mathcal{S} = (X, R, S, \mu)$, for all $x, y \in X$, if $xRy$ then let $\Pi_{\mathcal{S}}(x, y)$ be the set of all sequences $z_0, \ldots, z_n \in X$ such that $xRz_0 \ldots z_nRy$. Why are subordination structures so interesting? The following proposition contains a fact which helps to prove the starting point of our enterprise: $L$ is complete with respect to the class of all subordination structures of cardinal 1.

**Proposition 6.7** *Let $\phi \in f(BV)$. If $\phi$ is true in the class of all subordination structures of cardinal* 1 *then $\phi \in L$.*

**Proof.** Suppose $\phi \notin L$. Hence, by Lemma 6.3, $L + \neg\phi$ is a consistent $L$-theory. Thus, by Lemma 6.4, there exists $\Gamma \in MCT_L$ such that $L + \neg\phi \subseteq \Gamma$. Thus, $\neg\phi \in \Gamma$. Since $\Gamma$ is consistent, $\phi \notin \Gamma$. Let $\mathcal{S} = (X, R, S, \mu)$ be the structure such that $X = \{0\}$, $R = \emptyset$, $S = \emptyset$ and $\mu$ is the function $\mu$: $X \rightarrow MCT_L$ such that $\mu(0) = \Gamma$. Obviously, $\mathcal{S}$ is a subordination structure of cardinal 1 such that $\phi \notin \mu(0)$. Therefore, $\phi$ is not true in the class of all subordination structures of cardinal 1. $\Box$

It follows from Proposition 6.7 that we have reduced the task of proving the completeness of $L$ with respect to the class of all frames to the task of showing how to transform any subordination structure of cardinal 1 into a model satisfying the same formulas. One remark is in order here. Given a subordination structure $\mathcal{S} = (X, R, S, \mu)$, it may contain imperfections:

- $\Box$-*imperfections*, i.e. triples of the form $(x, \Box, \phi)$ where $x \in X$ and $\phi \in f(BV)$ are such that $\Box\phi \notin \mu(x)$ and for all $y \in X$, if $xRy$ then $\phi \in \mu(y)$,
- $\Box^\star$-*imperfections*, i.e. triples of the form $(x, \Box^\star, \phi)$ where $x \in X$ and $\phi \in f(BV)$ are such that $\Box^\star\phi \notin \mu(x)$ and for all $y \in X$, if $xSy$ then $\phi \in \mu(y)$,
- *imperfections of density*, i.e. pairs of the form $(x, y)$ where $x, y \in X$ are such that $xSy$ and for all $z \in X$, not $xRz$ or not $zSy$.

Remark that for all subordination structures $\mathcal{S} = (X, R, S, \mu)$, the imperfections of $\mathcal{S}$ are elements of $(\mathbb{Z} \times \{\Box, \Box^\star\} \times f(BV)) \cup (\mathbb{Z} \times \mathbb{Z})$.

## 6.2 Repairing imperfections

Lemmas 6.8, 6.10 and 6.12 state that every imperfection can be repaired.

**Lemma 6.8** *Let $\mathcal{S} = (X, R, S, \mu)$ be a subordination structure and $(x, \Box, \phi)$ be a $\Box$-imperfection in $\mathcal{S}$. There exists a subordination structure $\mathcal{S}' = (X', R', S', \mu')$ such that $\mathcal{S} \ll \mathcal{S}'$ and $(x, \Box, \phi)$ is not a $\Box$-imperfection in $\mathcal{S}'$.*

**Proof.** Since $(x, \Box, \phi)$ is a $\Box$-imperfection in $\mathcal{S}$, $x \in X$ and $\phi \in f(BV)$ are such that $\Box\phi \notin \mu(x)$ and for all $y \in X$, if $xRy$ then $\phi \in \mu(y)$. Since $\Box\phi \notin \mu(x)$, by Lemma 6.5, there exists $\Gamma \in MCT_L$ such that $\Box\mu(x) \subseteq \Gamma$ and $\phi \notin \Gamma$. Let $y \in \mathbb{Z} \setminus X$. We define the structure $\mathcal{S}' = (X', R', S', \mu')$ as follows:

- $X' = X \cup \{y\}$,
- $R'$ is the binary relation on $X'$ such that for all $x', y' \in X'$, $x'R'y'$ iff one of the following conditions holds:
  - · $x', y' \in X$ and $x'Ry'$,
  - · $x' \in X$, $y' = y$ and $x'Rx$,
  - · $x' \in X$, $y' = y$ and $x' = x$,
- $S'$ is the binary relation on $X'$ such that for all $x', y' \in X'$, $x'S'y'$ iff one of the following conditions holds:
  - · $x', y' \in X$ and $x'Sy'$,
  - · $x' \in X$, $y' = y$ and $x'Sx$,

- $\mu'$ is the function $\mu'$: $X' \to MCT_L$ such that for all $x' \in X'$,
  · if $x' \in X$ then $\mu'(x') = \mu(x')$,
  · if $x' = y$ then $\mu'(x') = \Gamma$.

Obviously, $R'$ is a strict partial order on $X'$, $S'$ is a strict partial order on $X'$, $S' \subseteq R'$, $R' \circ S' \subseteq S'$ and $S' \circ R' \subseteq S'$. Moreover, for all $x', y' \in X'$, if $x'R'y'$ then $\Box\mu'(x') \subseteq \mu'(y')$ and for all $x', y' \in X'$, if $x'S'y'$ then $\Box^\star\mu'(x') \subseteq \mu'(y')$. Hence, $\mathcal{S}'$ is a subordination structure. In other respect, as the reader can check, $\mathcal{S} \ll \mathcal{S}'$ and $(x, \Box, \phi)$ is not a $\Box$-imperfection in $\mathcal{S}'$. $\Box$

**Remark 6.9** Note that for all $x', y' \in X$, $\Pi_{\mathcal{S}'}(x', y') = \Pi_{\mathcal{S}}(x', y')$.

**Lemma 6.10** *Let $\mathcal{S} = (X, R, S, \mu)$ be a subordination structure and $(x, \Box^\star, \phi)$ be a $\Box^\star$-imperfection in $\mathcal{S}$. There exists a subordination structure $\mathcal{S}' = (X', R', S', \mu')$ such that $\mathcal{S} \ll \mathcal{S}'$ and $(x, \Box^\star, \phi)$ is not a $\Box^\star$-imperfection in $\mathcal{S}'$.*

**Proof.** Since $(x, \phi)$ is a $\Box^\star$-imperfection in $\mathcal{S}$, $x \in X$ and $\phi \in f(BV)$ are such that $\Box^\star\phi \notin \mu(x)$ and for all $y \in X$, if $xSy$ then $\phi \in \mu(y)$. Since $\Box^\star\phi \notin \mu(x)$, by Lemma 6.5, there exists $\Gamma \in MCT_L$ such that $\Box^\star\mu(x) \subseteq \Gamma$ and $\phi \notin \Gamma$. Let $y \in \mathbb{Z} \setminus X$. We define the structure $\mathcal{S}' = (X', R', S', \mu')$ as follows:

- $X' = X \cup \{y\}$,
- $R'$ is the binary relation on $X'$ such that for all $x', y' \in X'$, $x'R'y'$ iff one of the following conditions holds:
  · $x', y' \in X$ and $x'Ry'$,
  · $x' \in X$, $y' = y$ and $x'Rx$,
  · $x' \in X$, $y' = y$ and $x' = x$,
- $S'$ is the binary relation on $X'$ such that for all $x', y' \in X'$, $x'S'y'$ iff one of the following conditions holds:
  · $x', y' \in X$ and $x'Sy'$,
  · $x' \in X$, $y' = y$ and $x'Rx$,
  · $x' \in X$, $y' = y$ and $x' = x$,
- $\mu'$ is the function $\mu'$: $X' \to MCT_L$ such that for all $x' \in X'$,
  · if $x' \in X$ then $\mu'(x') = \mu(x')$,
  · if $x' = y$ then $\mu'(x') = \Gamma$.

Obviously, $R'$ is a strict partial order on $X'$, $S'$ is a strict partial order on $X'$, $S' \subseteq R'$, $R' \circ S' \subseteq S'$ and $S' \circ R' \subseteq S'$. Moreover, for all $x', y' \in X'$, if $x'R'y'$ then $\Box\mu'(x') \subseteq \mu'(y')$ and for all $x', y' \in X'$, if $x'S'y'$ then $\Box^\star\mu'(x') \subseteq \mu'(y')$. Hence, $\mathcal{S}'$ is a subordination structure. In other respect, as the reader can check, $\mathcal{S} \ll \mathcal{S}'$ and $(x, \Box^\star, \phi)$ is not a $\Box^\star$-imperfection in $\mathcal{S}'$. $\Box$

**Remark 6.11** Note that for all $x', y' \in X$, $\Pi_{\mathcal{S}'}(x', y') = \Pi_{\mathcal{S}}(x', y')$.

**Lemma 6.12** *Let $\mathcal{S} = (X, R, S, \mu)$ be a subordination structure and $(x, y)$ be an imperfection of density in $\mathcal{S}$. There exists a subordination structure $\mathcal{S}' = (X', R', S', \mu')$ such that $\mathcal{S} \ll \mathcal{S}'$ and $(x, y)$ is not an imperfection of density in $\mathcal{S}'$.*

**Proof.** Since $(x, y)$ is an imperfection of density in $\mathcal{S}$, $x, y \in X$ are such that $xSy$ and for all $z \in X$, not $xRz$ or not $zSy$. Since $xSy$, $\Box^\star\mu(x) \subseteq \mu(y)$. Hence, by Lemma 6.6, there exists $\Gamma \in MCT_L$ such that $\Box\mu(x) \subseteq \Gamma$ and $\Box^\star\Gamma \subseteq \mu(y)$. Let $z \in \mathbb{Z} \setminus X$. We define the structure $\mathcal{S}' = (X', R', S', \mu')$ as follows:

- $X' = X \cup \{z\}$,
- $R'$ is the binary relation on $X'$ such that for all $x', y' \in X'$, $x'R'y'$ iff one of the following conditions holds:
  - · $x', y' \in X$ and $x'Ry'$,
  - · $x' \in X$, $y' = z$ and $x'Rx$,
  - · $x' \in X$, $y' = z$ and $x' = x$,
  - · $x' = z$, $y' \in X$ and $yRy'$,
  - · $x' = z$, $y' \in X$ and $y' = y$,
- $S'$ is the binary relation on $X'$ such that for all $x', y' \in X'$, $x'S'y'$ iff one of the following conditions holds:
  - · $x', y' \in X$ and $x'Sy'$,
  - · $x' \in X$, $y' = z$ and $x'Sx$,
  - · $x' = z$, $y' \in X$ and $yRy'$,
  - · $x' = z$, $y' \in X$ and $y' = y$,
- $\mu'$ is the function $\mu'$: $X' \to MCT_L$ such that for all $x' \in X'$,
  - · if $x' \in X$ then $\mu'(x') = \mu(x')$,
  - · if $x' = z$ then $\mu'(x') = \Gamma$.

Obviously, $R'$ is a strict partial order on $X'$, $S'$ is a strict partial order on $X'$, $S' \subseteq R'$, $R' \circ S' \subseteq S'$ and $S' \circ R' \subseteq S'$. Moreover, for all $x', y' \in X'$, if $x'R'y'$ then $\Box\mu'(x') \subseteq \mu'(y')$ and for all $x', y' \in X'$, if $x'S'y'$ then $\Box^\star\mu'(x') \subseteq \mu'(y')$. Thus, $\mathcal{S}'$ is a subordination structure. In other respect, as the reader can check, $\mathcal{S} \ll \mathcal{S}'$ and $(x, y)$ is not an imperfection of density in $\mathcal{S}'$. $\Box$

**Remark 6.13** Note that for all $x', y' \in X$, $x'S'y'$ or $\Pi_{\mathcal{S}'}(x', y') = \Pi_{\mathcal{S}}(x', y')$.

Let the structures defined in the proofs of Lemmas 6.8, 6.10 and 6.12 be respectively called *completion* of $\mathcal{S}$ with respect to $(x, \Box, \phi)$, *completion* of $\mathcal{S}$ with respect to $(x, \Box^\star, \phi)$ and *completion* of $\mathcal{S}$ with respect to $(x, y)$.

### 6.3 Completeness

The following proposition constitutes the heart of our method.

**Proposition 6.14** *Let $\phi \in f(BV)$. If $\phi$ is valid in the class of all frames then $\phi$ is true in the class of all subordination structures of cardinal* 1.

**Proof.** Suppose $\phi$ is not true in the class of all subordination structures of cardinal 1. Hence, there exists a subordination structure $\mathcal{S} = (X, R, S, \mu)$ of cardinal 1 such that $\mathcal{S} \not\models \phi$. Let $i_0$, $i_1$, ... be an enumeration of $(\mathbb{Z} \times \{\Box, \Box^\star\} \times f(BV)) \cup (\mathbb{Z} \times \mathbb{Z})$ where each item is repeated infinitely often. We inductively define the sequence $\mathcal{S}_0 = (X_0, R_0, S_0, \mu_0)$, $\mathcal{S}_1 = (X_1, R_1, S_1, \mu_1)$, ... of subordination structures as follows:

- let $\mathcal{S}_0$ be $\mathcal{S}$,

- for all nonnegative integers $n$, if $i_n$ is an imperfection in $\mathcal{S}_n$ then let $\mathcal{S}_{n+1}$ be the completion of $\mathcal{S}_n$ with respect to $i_n$ else let $\mathcal{S}_{n+1}$ be $\mathcal{S}_n$.

Let $\mathcal{M}' = (X', R', S', V')$ be the structure defined as follows: $X' = \bigcup\{X_n\colon n$ is a nonnegative integer$\}$, $R' = \bigcup\{R_n\colon n$ is a nonnegative integer$\}$, $S' = \bigcup\{S_n\colon$ $n$ is a nonnegative integer$\}$ and $V'$ is the function $V'\colon BV \to \mathcal{P}(X)$ such that for all $p \in BV$, $V'(p) = \{x'\colon$ there exists a nonnegative integer $n$ such that $x' \in X_n$ and $p \in \mu_n(x')\}$. Obviously, $R'$ is a strict partial order on $X'$, $S'$ is a strict partial order on $X'$, $S' \subseteq R'$, $R' \circ S' = S'$ and $S' \circ R' \subseteq S'$. Hence, $S'$ is a fixpoint of the $\theta$-like function defined by $R'$. Now, let $S''$ be a fixpoint of the $\theta$-like function defined by $R'$, we demonstrate $S'' \leq S'$. Let $x', y' \in X'$ be such that $s'S''y'$, we demonstrate $x'S'y'$. Since $S''$ is a fixpoint of the $\theta$-like function defined by $R'$, $R' \circ S'' = S''$. Since $x'S''y'$, we can inductively construct an infinite sequence $z'_0, z'_1, \ldots \in X'$ such that $x'R'z'_0R'z'_1 \ldots$ and for all nonnegative integers $n$, $z'_nS''y'$. By Remarks 6.9, 6.11 and 6.13, there exists a nonnegative integer $n$ such that $x', y' \in X_n$ and $x'S_ny'$. Thus, $x'S'y'$. In conclusion, we have proved that

**Claim 6.15** *$S'$ is the greatest fixpoint of the $\theta$-like function defined by $R'$.*

Moreover, for all $x', y' \in X'$, if $x'R'y'$ then $\Box\mu'(x') \subseteq \mu'(y')$ and for all $x', y' \in X'$, if $x'S'y'$ then $\Box^\star\mu'(x') \subseteq \mu'(y')$. Now, let $\psi \in f(BV)$, we prove for all $x' \in X'$, $\mathcal{M}', x' \models \psi$ iff there exists a nonnegative integer $n$ such that $x' \in X_n$ and $\psi \in \mu_n(x')$. The proof is done by induction on $\psi$.
**Induction hypothesis.** Let $\psi \in f(BV)$ be such that for all $\chi \in f(BV)$, if $\chi$ is a subformula of $\psi$ then for all $x' \in X'$, $\mathcal{M}', x' \models \chi$ iff there exists a nonnegative integer $n$ such that $x' \in X_n$ and $\chi \in \mu_n(x')$.
**Induction step.** We have to consider the six following cases.
**Case $\psi = p$.** By definition of $V'$.
**Cases $\psi = \bot$, $\psi = \neg\chi$, $\psi = \chi' \vee \chi''$.** By the induction hypothesis.
**Cases $\psi = \Box\chi$, $\psi = \Box^\star\chi$.** By the induction hypothesis, by the fact that for all $x', y' \in X'$, if $x'R'y'$ then $\Box\mu'(x') \subseteq \mu'(y')$, by the fact that for all $x', y' \in X'$, if $x'S'y'$ then $\Box^\star\mu'(x') \subseteq \mu'(y')$, by the fact that for all $x' \in X'$ if $\Box\chi \notin \mu'(x')$ then there exists $y' \in X'$ such that $x'R'y'$ and $\chi \notin \mu'(y')$ and by the fact that for all $x' \in X'$ if $\Box^\star\chi \notin \mu'(x')$ then there exists $y' \in X'$ such that $x'S'y'$ and $\chi \notin \mu'(y')$.
In conclusion, we have proved that

**Claim 6.16** *Let $\psi \in f(BV)$. For all $x' \in X'$, $\mathcal{M}', x' \models \psi$ iff there exists a nonnegative integer $n$ such that $x' \in X_n$ and $\psi \in \mu_n(x')$.*

Since $\mathcal{S} \not\models \phi$, $\phi \notin \mu_0(0)$. By the above claim, $\mathcal{M}', 0 \not\models \phi$. Therefore, $\phi$ is not valid in the class of all frames. □

The result that emerges from the above discussion is the following theorem.

**Theorem 6.17** *Let $\phi \in f(BV)$. The following conditions are equivalent: (i) $\phi \in L$, (ii) $\phi$ is valid in the class of all frames, (iii) $\phi$ is true in the class of all subordination structures of cardinal* 1.

**Proof.** (i)$\rightarrow$(ii): By Proposition 6.1.
(ii)$\rightarrow$(iii): By Proposition 6.14.
(iii)$\rightarrow$(i): By Proposition 6.7. $\square$

## 7 Definability

In this section, we show that $\square^\star$ is not definable in the ordinary language of modal logic and that the class of all frames is not first-order definable.

### 7.1 Modal definability

Suppose there exists a $\square^\star$-free formula $\phi$ such that $\square^\star p \leftrightarrow \phi \in L$. Let $\mathcal{M} = (\mathbb{Z}, <_{\mathbb{Z}}, \emptyset, V)$ be the model defined over the integers and such that for all $q \in BV$, $V(q) = \emptyset$ and $\mathcal{M}' = (\mathbb{Q}, <_{\mathbb{Q}}, <_{\mathbb{Q}}, V')$ be the model defined over the rationals and such that for all $q \in BV$, $V'(q) = \emptyset$. Obviously, for all $\square^\star$-free formulas $\psi$, for all $x \in \mathbb{Z}$ and for all $x' \in \mathbb{Q}$, $\mathcal{M}, x \models \psi$ iff $\mathcal{M}', x' \models \psi$. Hence, $\mathcal{M}, 0 \models \phi$ iff $\mathcal{M}', 0 \models \phi$ Since $S_{\mathcal{M}} = \emptyset$, $\mathcal{M}, 0 \models \square^\star p$. Since $\square^\star p \leftrightarrow \phi \in L$, by Proposition 6.1, $\mathcal{M}, 0 \models \phi$. Since $S_{\mathcal{M}'} = <_{\mathbb{Q}}$, $\mathcal{M}', 0 \not\models \square^\star p$. Since $\square^\star p \leftrightarrow \phi \in L$, by Proposition 6.1, $\mathcal{M}', 0 \not\models \phi$: a contradiction. These considerations prove

**Proposition 7.1** *There exists no $\square^\star$-free formula $\phi$ such that $\square^\star p \leftrightarrow \phi \in L$.*

That is to say, $\square^\star$ is not definable in the ordinary language of modal logic.

### 7.2 First-order definability

Suppose there exists a first-order sentence $\phi$ in $\tilde{R}$, $\tilde{S}$ and $\equiv$ (interpreted in a relational structure $\mathcal{F} = (X, R, S)$ by $R$, $S$ and equality) such that for all relational structures $\mathcal{F} = (X, R, S)$, $\mathcal{F}$ is a frame iff $\mathcal{F} \models \phi$. For all $n \in \mathbb{N}$, let $\mathcal{F}_n = (X_n, R_n, S_n)$ be the relational structure defined as follows: $X_n = \{0, \ldots, n\}$, $R_n = \{(i, j)\colon\ 0 \leq i < j \leq n\}$ and $S_n = \emptyset$, Obviously, for all $n \in \mathbb{N}$, $\mathcal{F}_n \models \phi \wedge \exists y \forall x (x\tilde{R}y \vee x \equiv y) \wedge \forall x \forall y \neg x\tilde{S}y$. Let $U$ be an ultrafilter over $\mathbb{N}$ and $\mathcal{F}_U = (X_U, R_U, S_U)$ be the ultraproduct of the family $\{\mathcal{F}_n\colon n \in \mathbb{N}\}$ modulo $U$. Since for all $n \in \mathbb{N}$, $\mathcal{F}_n \models \phi \wedge \exists y \forall x (x\tilde{R}y \vee x \equiv y) \wedge \forall x \forall y \neg x\tilde{S}y$, by the Fundamental Theorem of Ultraproducts [7, Theorem 4.1.9], $\mathcal{F}_U \models \phi \wedge \exists y \forall x (x\tilde{R}y \vee x \equiv y) \wedge \forall x \forall y \neg x\tilde{S}y$. Since $\mathcal{F}_U \models \phi$, $\mathcal{F}_U$ is a frame. For all $i \in \mathbb{N}$, let $[i]$ be the class of $(i, i, \ldots)$ modulo $U$. Remark that for all $i, j \in \mathbb{N}$, $[i] R_U [j]$ iff $i < j$. Since $\mathcal{F}_U \models \exists y \forall x (x\tilde{R}y \vee x \equiv y)$, there exists $M_U \in X_U$ such that for all $i \in \mathbb{N}$, $[i] R_U M_U$ or $[i] = M_U$. Since for all $i, j \in \mathbb{N}$, $[i] R_U [j]$ iff $i < j$, for all $i \in \mathbb{N}$, $[i] R_U M_U$. Let $R'_U$ be the binary relation on $X_U$ such that for all $x, y \in X_U$, $x R'_U y$ iff there exists $i \in \mathbb{N}$ such that $x = [i]$ and $y = M_U$, we demonstrate $R'_U \leq \theta_{R_U}(R'_U)$, i.e. $R'_U \leq R_U \circ R'_U$. Remark that $R'_U$ is a strict partial order on $X_U$ and $R'_U \subseteq R_U$. Moreover, $R'_U \neq \emptyset$. Let $x, y \in X_U$ be such that $x R'_U y$, we demonstrate there exists $z \in X_U$ such that $x R_U z$ and $z R'_U y$. Since $x R'_U y$, there exists $i \in \mathbb{N}$ such that $x = [i]$ and $y = M_U$. Hence, it suffices to take $z = [i+1]$ and we have $x R_U z$ and $z R'_U y$. In conclusion, we have proved that

**Claim 7.2** $R'_U \leq \theta_{R_U}(R'_U)$.

By the results stated in Section 4.2, $R'_U \leq \text{gfp}(\theta_{R_U})$. Since $\mathcal{F}_U \models \forall x \forall y \neg x \tilde{S} y$, $\text{gfp}(\theta_{R_U}) = \emptyset$. Since $R'_U \leq \text{gfp}(\theta_{R_U})$, $R'_U = \emptyset$: a contradiction. The conclusion can be summarized as follows.

**Proposition 7.3** *There exists no first-order sentence $\phi$ in $\tilde{R}$, $\tilde{S}$ and $\equiv$ (interpreted in a relational structure $\mathcal{F} = (X, R, S)$ by $R$, $S$ and equality) such that for all relational structures $\mathcal{F} = (X, R, S)$, $\mathcal{F}$ is a frame iff $\mathcal{F} \models \phi$.*

That is to say, the class of all frames is not first-order definable.

## 8 Conclusion

In this article, we considered a modal logic with modal operators $\Box$ and $\Box^\star$ respectively interpreted by strict partial orders and the greatest fixpoints of the $\theta$-like functions they define. Much remains to be done. Firstly, there is the issue of the complete axiomatization of the set of all formulas in the $\Box$-free fragment of our language that are valid in the class of all frames. Are the axioms of the form $\Box^\star\phi \to \Box^\star\Box^\star\phi$ sufficient in this respect? Secondly, there is the question of the computability and complexity of the membership problem in $L$. Obviously, $L$ is a conservative extension of $K4$. Hence, by Ladner's Theorem [4, Theorem 6.50], the membership problem in $L$ is $PSPACE$-hard. Is it possible to demonstrate that it is in $PSPACE$? Thirdly, there is the issue of the finite model property (fmp) of $L$. There are possibly two ways to ask whether $L$ has the fmp, depending on the class of relational structures one considers. One possibility is to consider the fmp with respect to the class of all frames. Another possibility is to consider the fmp with respect to the class of all relational structures satisfying the conditions considered in Lemma 5.1. Fourthly, there is the question of the modal definability of the class of all frames. More precisely, is there $\Gamma \subseteq f(BV)$ such that for all relational structures $\mathcal{F} = (X, R, S)$, $\mathcal{F}$ is a frame iff for all $\phi \in f(BV)$, if $\phi \in \Gamma$ then $\mathcal{F} \models \phi$? If such $\Gamma \subset f(BV)$ exists, can it be finite? Fifthly, there is the issue of the addition to our language of the global operator $[U]$ and the difference operator $[\neq]$ respectively interpreted by the universal relation and the inequality relation. As is well-known, see [4, Chapter 7] or [15], these modal operators greatly increase the expressive power of a modal language whether it is interpreted in relational structures as in Section 5.2 or in topological structures as in Section 5.3. Sixthly, there is the issue of the complete axiomatization of the set of all formulas that are valid in the class of all topological frames. Are the axioms of $L$ together with the axioms of the form $\Box(p \to \Diamond p) \to (\Diamond p \to \Diamond^\star p)$ sufficient in this respect? Seventhly, there is the question of the possible readings of $\Box^\star$ in terms of knowledge and belief. See [2,22] for more details.

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