Labelled sequent calculi for logics of strict implication

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Abstract

We study the proof theory of C.I. Lewis' logics of strict conditional **S1**-**S5** and we propose the first modular and uniform presentation of C.I. Lewis' systems. In particular, for each logic **Sn** we present a labelled sequent calculus **G3Sn** and we discuss its structural properties: every rule is height-preserving invertible and the structural rules of weakening, contraction and cut are admissible. Completeness of **G3Sn** is established both indirectly via the embedding in the axiomatic system **Sn** and directly via the extraction of a countermodel out of a failed proof search. Finally, the sequent calculus **G3S1** is employed to obtain a syntactic proof of decidability of **S1**.

Keywords: Strict implication, non-normal modalities, S1, sequent calculi cut elimination.

1 Introduction

Clarence Irving Lewis proposed the first axiomatic systems of propositional modal logic [7,8]. In particular, due to his dissatisfaction towards the material conception of classical implication, he devised a new logical operator, namely *strict implication* ($\strictif$). He introduced five systems from **S1** to **S5** [8]. The modal logics **S4** and **S5** have been intensively studied, whereas **S2**, **S3** and, above all, **S1** did not receive much attention.

It can be argued that this depended on the fact that the latter are non-normal modal logics, as the rule of necessitation does not hold unrestrictedly. The semantics of the systems **S2** and **S3** was obtained via a slight modification of the standard Kripke semantics, by considering models with non-normal (or queer) worlds [6].

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On the contrary, Cresswell [3] proposed a semantics for **S1** which combines features of neighborhood and relational models. [3] Due to the rather complex formulation of the semantics **S1** was long considered as an uninteresting system, but see [2]. In our opinion, this position is not justified, insofar as the system exhibits some interesting metalogical properties. In particular, the system is decidable and the modal operator defined from the strict implication has some hyperintensional features: unrestricted substitution of materially equivalent formulas does not hold, cf. [1,5,12].

In the present work we shall focus on the proof theory of these systems. In a previous paper by one of the two authors labelled sequent calculi were introduced for the logics **S2**, **S3** and some related systems [14]. However, a modular and uniform treatment is still lacking, due to the impossibility to encompass the system **S1**.

We propose the first modular and analytic approach to the proof theory of the original systems by C. I. Lewis (related systems are omitted for brevity). As in the tradition of labelled systems, sequent calculi are obtained by converting the truth conditions for the logical operators in corresponding rules. The rules introduced in Table 1 for $\prec$ correspond more directly to a simplification of Cresswell's semantics where the neighborhood function is replaced by a bi-neighborhood as it is done in [4] for non-normal modal logics. In bi-neighborhood semantics worlds are mapped to pairs of disjoint sets of worlds, providing 'independent positive and negative evidence (or support) for a proposition' [4, p. 161]. Neverthteless, this paper sticks to Cresswell's semantics for simplicity.

The calculi satisfy good structural properties, namely admissibility of the rules of weakening, contraction and cut, as well as invertibility of all rules.

Completeness is first established by showing the embedding of Lewis' axiomatic calculi into the corresponding labelled sequent calculi. The admissibility of the rule of substitution of strict equivalents requires to prove a non trivial lemma, see Lemma 4.9. We then establish a more direct form of completeness via the extraction of a countermodel out of a failed proof search and we discuss the relation between the **S1** neighborhood semantics and the bi-neighborhood framework.

Our proof-theoretic approach enables us to investigate the system **S1** by purely syntactic means which are uniform with respect to the ones traditionally employed for **S2** – **S5**. In particular, we exploit the calculus **G3S1** to obtain the first purely syntactic proof of decidability of the logic **S1** via terminating proof search. The decidability of the system **S1** had already been established by semantic means. In particular, the proof used the filtration method to prove the finite model property, see [2,3]. We are not aware of a syntactic proof of decidability for **S1**. This depends on the fact that axiomatic presentations are not amenable to proof search due to their substantial lack of analyticity.

[3] A semantics for **S1** based on Rantala models has been given in [15], and a relational semantics has been given in [13].

The structure of the paper is as follows. Section 2 recalls the logics **S1** – **S5**. Then, Section 3 introduces the labelled calculi **G3S1** – **G3S5** and Section 4 studies their structural properties. Section 5 gives a direct and modular proof of completeness for **G3Sn**. Finally Section 6 proves the decidability of **G3S1**.

2 Logics of strict implication

Language

The language of strict implications is defined by the following grammar:

$$A \;\;::=\;\; p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid A \strictif A \qquad (\mathcal{L}^{\strictif})$$

where $p \in \mathcal{P}$ for a denumerable set of sentential variables $\mathcal{P}$.

Parentheses are used as customary ($\strictif$ binds lighter than other operators). Capital roman letters will be used for arbitrary formulas and lower-case ones for sentential variables. We use $\equiv$ to denote syntactic identity. The symbol $\top$ is short for $\bot \supset \bot$ and $\neg A$ is short for $A \supset \bot$. The unary modalities $\Box$ and $\Diamond$ can be defined as: $\Box A \;\equiv\; \top \strictif A$ and $\Diamond A \;\equiv \neg(\top \strictif \neg A)$

We use $\mathcal{L}^{\Box}$ to denote the standard modal language—i.e., $\mathcal{L}^{\strictif}$ with $\Box$ and $\Diamond$ in place of $\strictif$. The formula $A \strictif B$ can be defined in $\mathcal{L}^{\Box}$ either as $\Box(A \supset B)$ or as $\neg\Diamond(A \wedge \neg B)$. Observe that languages $\mathcal{L}^{\strictif}$ and language $\mathcal{L}^{\Box}$ are not minimal since we have the usual classical and modal interdefinabilities—e.g., C.I. Lewis [8] considered a language with only $\neg$, $\wedge$ and $\Diamond$ as primitives.

We will use $A[B//C]$ for the formula obtained from A by replacing some occurrences of C with occurrences of B.

Axiomatic systems

We present here C.I. Lewis [8] axiomatisation of the logics **S1** – **S5**. As already anticipated Lewis considered a language with only $\neg$, $\wedge$ and $\Diamond$ as primitives. For simplicity we assume to have the definition of the other symbols as implicit axioms. We simplify Lewis' axiomatisations by dropping the redundant axiom $A \strictif \neg\neg A$—see [9]—and by considering axiom schemes instead of having as primitive a rule of uniform substitution of material equivalents.

Definition 2.1 [Lewis' axiomatisation of **S1**]

- Axioms:
  - (i) $(A \wedge B) \strictif (B \wedge A)$
  - (ii) $(A \wedge B) \strictif A$
  - (iii) $A \strictif (A \wedge A)$
  - (iv) $((A \wedge B) \wedge C) \strictif (A \wedge (B \wedge C))$
  - (v) $((A \strictif B) \wedge (B \strictif C)) \strictif (A \strictif C)$
  - (vi) $(A \wedge (A \strictif B)) \strictif B$
- Rules:
  - (i) $$\frac{A \quad (B \strictif C) \wedge (C \strictif B)}{A[B//C]}\ SSE$$
  - (ii) $$\frac{A \quad B}{A \wedge B}\ Adj$$
  - (iii) $$\frac{A \strictif B \quad A}{B}\ MP_{\strictif}$$

Definition 2.2 [Axiomatisation of **S2**–**S5**] **S2** = **S1** $\oplus\, \Diamond(A \wedge B) \strictif \Diamond A$; **S3** = **S2** $\oplus\, (A \strictif B) \strictif (\Box A \strictif \Box B)$; **S4** = **S1** $\oplus\, \Box A \strictif \Box\Box A$; **S5** = **S4** $\oplus\, A \strictif \Box\Diamond A$.

Semantics

As it is well-known, standard relational semantics can be used for the normal conditional logics **S4** and **S5**. A modification thereof has been used by Kripke [6] to give a semantics for the non-normal conditional logics **S2** and **S3**: we must add so-called *queer* (or *non-normal*) *worlds* where $\Diamond A$ is always true and $\Box A$ is always false. Finally, a semantics for **S1** has been introduced by Cresswell in [3] and generalised to logics weaker than **S1** in [2]. This semantics is interesting because it needs both an accessibility relation and a neighborhood functions to define strict implication (as well as $\Box$ and $\Diamond$): the accessibility relation is used in normal worlds and the neighborhood function is used in queer ones.

Formally an **S1**-*frame* is quadruple $\mathcal{F} = \langle \mathcal{W}, \mathcal{N}, \mathcal{R}, \mathcal{I} \rangle$ where: (i) $\mathcal{W}$ is a non-empty set of *worlds*; (ii) $\mathcal{N}$ is a subset of $\mathcal{W}$, of so-called *normal worlds* (worlds in $\mathcal{W} \backslash \mathcal{N}$ will be called *queer worlds*); (iii) $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ is a reflexive *accessibility relation* on $\mathcal{W}$; (iv) $\mathcal{I} : \mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ is a *neighborhood functions* mapping worlds to sets of sets of worlds with the side conditions that if $\alpha \in \mathcal{I}(w)$ then $w \in \alpha$—i.e., $\mathcal{I}$ is reflexive— and that if $X, Y \in I(w)$ then $X \cup Y \neq \mathcal{W}$.

By adding conditions on $\mathcal{N}, \mathcal{R}$, and $\mathcal{I}$ we can define a class of frames for the other Lewis' systems. In particular: (i) an **S2**-*frame* is an **S1**-frame where $\mathcal{I}$ is such that it maps each world to $\emptyset$; (ii) an **S3**-*frame* is a transitive **S2**-frame—i.e., if $w\mathcal{R}v$ and $v\mathcal{R}u$ then $w\mathcal{R}u$; (iii) an **S4**-*frame* is an **S3**-frame where $\mathcal{N} = \mathcal{W}$; (iv) an **S5**-*frame* is a symmetric **S4**-frame—i.e., if $w\mathcal{R}v$ then $v\mathcal{R}w$. Some observations are in order. **S2**-frames can be equivalently defined by simply dropping $\mathcal{I}$ from **S1** fames, thus obtaining Kripke semantics for non-normal logics [6]. **S4** can be defined by dropping $\mathcal{N}$ and $\mathcal{I}$ from **S1**-frames, thus obtaining standard relational semantics for normal modalities.

A *model* $\mathcal{M}$ is a frame augmented with a *valuation function* $\mathcal{V} : \mathcal{W} \longrightarrow \mathcal{P}(\mathcal{W})$ mapping each sentential variable to the set of worlds where it holds. We say that $\mathcal{M}$ is an **Sn**-*model* if its underlying frame is an **Sn**-frame.

We are now ready to define *truth* of a formula A at a world w of a model $\mathcal{M}$, $\models^{\mathcal{M}}_w A$ or simply $\models_w A$ when $\mathcal{M}$ is clear from the context. The definition is standard for sentential variables and for the extensional operators—e.g., $\models_w p$ iff $w \in \mathcal{V}(p)$ and $\models_w A \wedge B$ iff $\models_w A$ and $\models_w B$. The only interesting case is that of strict implication where we have:

$$\models_w A \prec B \qquad \text{iff} \qquad \begin{cases} \forall v \in \mathcal{W}, w\mathcal{R}v \text{ and } \models_v A \text{ imply } \models_v B, & \text{if } w \in \mathcal{N} \\ [\![A \supset B]\!]_{\mathcal{M}} \in \mathcal{I}(w), & \text{else} \end{cases}$$

where $[\![A]\!]_{\mathcal{M}}$ is the truth set of A in $\mathcal{M}$: $[\![A]\!]_{\mathcal{M}} = \{w : \models^{\mathcal{M}}_w A\}$. Equivalently, we have that $\models_w A \prec B$ for $w \in \mathcal{W} \setminus \mathcal{N}$ iff $\exists \alpha \in I(w)$ such that, for all $v \in \mathcal{W}$, ($\not\models_v A$ or $\models_v B$) if and only if $v \in \alpha$.

We now introduce two abbreviations. $\alpha \Vdash B$ expresses $\forall u(u \in \alpha \Rightarrow \models_u B)$ —i.e., every world in α satisfies the formula B. $\alpha \lhd B$ is the *covering relation* which asserts $\forall u(\models_u B \Rightarrow u \in \alpha)$—i.e., every world which satisfies B is in α. The latter can also be equivalently formulated as $\forall u \neg(u \notin \alpha \;\&\; \models_u B)$. The expression $[\![A \supset B]\!]_{\mathcal{M}} \in \mathcal{I}(w)$ can be rewritten as:

$$\exists \alpha \in \mathcal{I}(w)(\alpha \Vdash A \supset B \;\&\; \alpha \lhd A \supset B)$$

Observe that for **S2**- and **S3**-models the clause for queer worlds says that $A \strictif B$ cannot be true therein, and for **S4**- and **S5**-models it can be dropped.

A formula A is said to be: (i) *True in a model* $\mathcal{M}$, $\models^{\mathcal{M}} A$, if it true in every normal point of that model; (ii) **Sn**-*valid*, $\mathbf{Sn} \models A$, if it is true in all **Sn**-models; (iii) *An* **Sn**-*consequence* of a set of formulas X, $X \models_{\mathbf{Sn}} A$, if A is true in all normal world of each **Sn**-model where all formulas in X are true.

Theorem 2.3 (Characterisation, [3]) *The axiomatic calculus* **Sn** *is sound and compete for validity w.r.t. the class of all* **Sn**-*frames.*

3 Labelled sequent calculi

We are now going to introduce labelled sequent calculi for the logics of strict implication **S1**-**S5**. Labelled calculi for normal modal logics have been introduced in [10] and for the non-normal ones in [4,11]. Labelled calculi for the non-normal logics **S2** and **S3**, as well as for some of their extensions, based on the language $\mathcal{L}^{\Box}$ have been studied in [14]. Here we consider also **S1** and we work with a language with $\strictif$ instead of $\Box$ as primitive.

To define the language of sequent calculi we consider two denumerable sets of labels: a set $\mathbb{W}$ of *world labels*, for which we use the metavariables $w, v, u, \dots$, and a set $\mathbb{I}$ of *neighbours label*, denoted by $\alpha, \beta, \gamma, \dots$. Moreover, we add the following atomic predicates $R, N, Q, \in$, and $\notin$ that are syntactic counterparts of the elements of **S1**-frames. The formulas of the labelled language $\mathcal{L}^{ll}$ are the following (where $w, v \in \mathbb{W}$, $\alpha \in \mathbb{I}$ and $A \in \mathcal{L}^{\strictif}$): (i) *relational atoms* wRv; (ii) *normality atoms* Nw; (iii) *queer atoms* Qw; (iv) *neighbour atoms* $\alpha \in Iw$; (v) *inclusion atoms* $w \in \alpha$; (vi) *exclusion atoms* $w \notin \alpha$; (vii) *labelled formulas* $w : A$; (viii) *forcing formulas* $\alpha \Vdash A$; and (ix) *covering formulas* $\alpha \lhd A$.

Definition 3.1 The *label* of a formula E in $\mathcal{L}^{ll}$ of form $u : A$ (resp. $\alpha \Vdash A$ or $\alpha \lhd A$) is u (resp. α) and is denoted by $l(E)$. The *pure part* of a labelled formula E is obtained removing from E the label and the forcing and is denoted by $p(E)$. The notion of *weight* is defined for labels and pure parts of formulas. For every $u \in \mathbb{W}$, $\mathtt{w}(u) = 0$, and for every $a \in \mathbb{I}$, $\mathtt{w}(\alpha) = 1$. The weight of a pure formula A, $\mathtt{w}(A)$ is defined as follows: $\mathtt{w}(p) = \mathtt{w}(\bot) = 1$, $\mathtt{w}(A \circ B) = max(\{\mathtt{w}(A), \mathtt{w}(B)\}) + 1$, where $\circ \in \{\wedge, \vee, \supset\}$, $\mathtt{w}(A \strictif B) = max(\{\mathtt{w}(A), \mathtt{w}(B)\}) + 2$. The *degree* of a labelled, forcing, or covering formula E is an ordered pair $\mathtt{deg}(E) = (\mathtt{w}(p(E)), \mathtt{w}(l(E)))$. Additionally, we stipulate $\mathtt{deg}(wRu) = \mathtt{deg}(Nu) = \mathtt{deg}(Qu) = \mathtt{deg}(\alpha \in Iu) = \mathtt{deg}(u \in \alpha) = \mathtt{deg}(u \notin \alpha) = (0, 1)$. *Degrees* of $\mathcal{L}^{ll}$-formulas are ordered lexicographically.

A *sequent* is an expression $\Gamma \Rightarrow \Delta$ where Γ is a finite multiset of $\mathcal{L}^{ll}$-formulas and Δ is a finite multiset of labelled, forcing, and covering formulas only. Substitutions of labels in an $\mathcal{L}^{ll}$-formula E, $E[v/u]$ and $E[\alpha/\beta]$, are defined as expected and it is extended to multisets by applying it componentwise.

The rules of the calculi **G3S1**–**G3S5** are given in Table 1: **G3S1** contains all initial sequent and all propositional, conditional, and relational rules. **G3S2** = **G3S1** plus rule $S2$. **G3S3** = **G3S2** plus rule *Trans*. **G3S4** = **G3S3** plus rule *Norm*. **G3S5** = **G3S4** plus rule *Sym*. Observe that the calculus

G3S2 (**G3S4**) is equivalent to the simpler calculus obtained by dropping rules $L/R \dashv_Q$ (and removing normality atoms from rules $L/R \dashv_N$) and all relational rules but Ref_R from the calculus **G3S1** (**G3S3**).

A **G3Sn**-*derivation* of a sequent $\Gamma \Rightarrow \Delta$ is a tree of sequents, whose leaves are initial sequents, whose root is $\Gamma \Rightarrow \Delta$, and which grows according to the rules of **G3Sn**. The *height* of a **G3Sn**-derivation is the number of nodes of a branch of maximal length. We say that $\Gamma \Rightarrow \Delta$ is **G3Sn**-derivable (with height n), and we write **G3Sn** $\vdash^{(n)} \Gamma \Rightarrow \Delta$, if there is a **G3Sn**-derivation (of height at most n) of $\Gamma \Rightarrow \Delta$. A rule is said to be *(height-preserving) admissible* in **G3Sn**, if, whenever its premisses are **G3Sn**-derivable (with height at most n), also its conclusion is **G3Sn**-derivable (with height at most n). In each rule depicted in Table 1, Γ and Δ are called *contexts*, the formulas occurring in the conclusion are called *principal*, and those occurring in the premisses only are called *active*.

Lemma 3.2 *The sequent* $E, \Gamma \Rightarrow \Delta, E$ *is* **G3Sn**-*derivable for every* Γ, Δ, E.

Proof By induction on the degree of the formula E: the rules are applied root-first since in each branch we reach a sequent with a formula occurring both in the antecedent and in the succedent and having lesser degree than E.□

4 Structural properties

Lemma 4.1 (Substitution) **G3Sn** $\vdash^n \Gamma \Rightarrow \Delta$ *implies* **G3Sn** $\vdash^n \Gamma[v/u] \Rightarrow \Delta[v/u]$ *and* **G3Sn** $\vdash^n \Gamma[\alpha/\beta] \Rightarrow \Delta[\alpha/\beta]$.

Proof A standard induction on the height of the derivation $\mathcal{D}$ of the sequent $\Gamma \Rightarrow \Delta$. We apply to $\mathcal{D}$ the inductive hypothesis either twice or once— depending on whether the last rule instance *Rule* in $\mathcal{D}$ has a variable condition that clashes with the substitution or not— and then we conclude by applying an instance of *Rule*. □

Theorem 4.2 (Weakening) *For every multiset* Π *and* Σ, **G3Sn** $\vdash^n \Gamma \Rightarrow \Delta$ *implies* **G3Sn** $\vdash^n \Pi, \Gamma \Rightarrow \Delta, \Sigma$.

Proof By induction on the height of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$, possibly applying an (hp-admissible) instance of substitution if the last rule instance in $\mathcal{D}$ has a variable condition. □

Lemma 4.3 *If* A *is an axiom of the axiomatic system* **Sn** *then the sequent* $Nw \Rightarrow w : A$ *is* **G3Sn**-*derivable.*

Proof The proof is straightforward by a root-first application of the rules of the calculi, possibly using the admissiblity of weakening. We limit ourselves to considering axiom (v).

Table 1
Rules of the calculi **G3S1**–**G3S5**

Initial Sequents	$\dfrac{}{w : p, \Gamma \Rightarrow \Delta, w : p}\, {\scriptstyle Ax}$	$\dfrac{}{w : \bot, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\bot}$
	$\dfrac{}{Nw, Qw, \Gamma \Rightarrow \Delta}\, {\scriptstyle Ax_N}$	$\dfrac{}{w \in \alpha, w \notin \alpha, \Gamma \Rightarrow \Delta}\, {\scriptstyle Ax_\in}$
Propositional Rules		
$\dfrac{w : A, w : B, \Gamma \Rightarrow \Delta}{w : A \wedge B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\wedge}$	$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \wedge B}\, {\scriptstyle R\wedge}$	$\dfrac{w : A, \Gamma \Rightarrow \Delta \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\vee}$
$\dfrac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B}\, {\scriptstyle R\vee}$	$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \supset B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\supset}$	$\dfrac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \supset B}\, {\scriptstyle R\supset}$
Conditional Rules	$\dfrac{Nw, wRv, w : A \mathrel{-\!\!3} B, \Gamma \Rightarrow \Delta, v : A \qquad v : B, Nw, wRv, w : A \mathrel{-\!\!3} B, \Gamma \Rightarrow \Delta}{Nw, wRv, w : A \mathrel{-\!\!3} B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L \mathrel{-\!\!3}_N}$	
$\dfrac{u : A, wRu, Nw, \Gamma \Rightarrow \Delta, u : B}{Nw, \Gamma \Rightarrow \Delta, w : A \mathrel{-\!\!3} B}\, {\scriptstyle R \mathrel{-\!\!3}_N,\ u \text{ fresh}}$	$\dfrac{\alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, \Gamma \Rightarrow \Delta}{Qw, w : A \mathrel{-\!\!3} B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L \mathrel{-\!\!3}_Q,\ \alpha \text{ fresh}}$	
$\dfrac{Qw, \alpha \in Iw, \Gamma \Rightarrow \Delta, w : A \mathrel{-\!\!3} B, \alpha \Vdash A \supset B \qquad Qw, \alpha \in Iw, \Gamma \Rightarrow \Delta, w : A \mathrel{-\!\!3} B, \alpha \lhd A \supset B}{Qw, \alpha \in Iw, \Gamma \Rightarrow \Delta, w : A \mathrel{-\!\!3} B}\, {\scriptstyle R \mathrel{-\!\!3}_Q}$		
Relational rules	$\dfrac{v : A, v \in \alpha, \alpha \Vdash A, \Gamma \Rightarrow \Delta}{v \in \alpha, \alpha \Vdash A, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\Vdash}$	$\dfrac{u \in \alpha, \Gamma \Rightarrow \Delta, u : A}{\Gamma \Rightarrow \Delta, \alpha \Vdash A}\, {\scriptstyle R\Vdash,\ u \text{ fresh}}$
$\dfrac{v \notin \alpha, \alpha \lhd A, \Gamma \Rightarrow \Delta, v : A}{v \notin \alpha, \alpha \lhd A, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\lhd}$	$\dfrac{u \notin \alpha, u : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha \lhd A}\, {\scriptstyle R\lhd,\ u \text{ fresh}}$	$\dfrac{wRw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, {\scriptstyle Ref_R}$
$\dfrac{u \notin \alpha, u \notin \beta, \alpha \in Iw, \beta \in Iw, \Gamma \Rightarrow \Delta}{\alpha \in Iw, \beta \in Iw, \Gamma \Rightarrow \Delta}\, {\scriptstyle S1,\ u \text{ fresh}}$	$\dfrac{w \in \alpha, \alpha \in Iw, \Gamma \Rightarrow \Delta}{\alpha \in Iw, \Gamma \Rightarrow \Delta}\, {\scriptstyle Ref_I}$	
$\dfrac{Nw, \Gamma \Rightarrow \Delta \qquad Qw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, {\scriptstyle Norm}$		
Additional rules	$\dfrac{}{\alpha \in Iw, \Gamma \Rightarrow \Delta}\, {\scriptstyle S2}$	$\dfrac{wRu, wRv, vRu, \Gamma \Rightarrow \Delta}{wRv, vRu, \Gamma \Rightarrow \Delta}\, {\scriptstyle Trans}$
	$\dfrac{Nw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, {\scriptstyle Norm}$	$\dfrac{uRw, wRu, \Gamma \Rightarrow \Delta}{wRu, \Gamma \Rightarrow \Delta}\, {\scriptstyle Sym}$

$$
\dfrac{\dfrac{\dfrac{Nw, Nu, wRu, u : A \mathrel{-\!\!3} B, u : B \mathrel{-\!\!3} C \Rightarrow u : A \mathrel{-\!\!3} C \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[...], v : B, v : A \Rightarrow v : B, v : C, [...]}\,{\scriptstyle \text{Lem.3.2}}}{[...], v : A \Rightarrow v : B, v : B \supset C, [...]}\,{\scriptstyle \text{R}\supset}}{[...] \Rightarrow v : A \supset B, v : B \supset C, [...]}\,{\scriptstyle \text{R}\supset}}{[...], v \notin \alpha, v \notin \beta, \alpha \lhd A \supset B, \beta \lhd B \supset C \Rightarrow u : A \mathrel{-\!\!3} C, v : A \supset B}\,{\scriptstyle \text{L}\lhd}}{[...], v \notin \alpha, v \notin \beta, \alpha \in Iu, \alpha \lhd A \supset B, \beta \in Iu, \beta \lhd B \supset C \Rightarrow u : A \mathrel{-\!\!3} C}\,{\scriptstyle \text{L}\lhd}}{[...], \alpha \in Iu, \alpha \lhd A \supset B, \beta \in Iu, \beta \Vdash B \supset C, \beta \lhd B \supset C \Rightarrow u : A \mathrel{-\!\!3} C}\,{\scriptstyle S1}}{[...], Qu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, u : B \mathrel{-\!\!3} C \Rightarrow u : A \mathrel{-\!\!3} C}\,{\scriptstyle \text{L}\mathrel{-\!\!3}_Q}}{Qu, u : A \mathrel{-\!\!3} B, u : B \mathrel{-\!\!3} C \Rightarrow u : A \mathrel{-\!\!3} C}\,{\scriptstyle \text{L}\mathrel{-\!\!3}_Q}}{Nw, wRu, u : A \mathrel{-\!\!3} B, u : B \mathrel{-\!\!3} C \Rightarrow u : A \mathrel{-\!\!3} C}\,{\scriptstyle Norm}}{Nw, wRu, u : (A \mathrel{-\!\!3} B) \wedge (B \mathrel{-\!\!3} C) \Rightarrow u : A \mathrel{-\!\!3} C}\,{\scriptstyle \text{L}\wedge}}{Nw \Rightarrow w : ((A \mathrel{-\!\!3} B) \wedge (B \mathrel{-\!\!3} C)) \mathrel{-\!\!3} (A \mathrel{-\!\!3} C)}\,{\scriptstyle \text{R}\mathrel{-\!\!3}_N}
$$

The leftmost top-sequent is provable via applications of rules R-$\mathrel{-\!\!3}_N$ and L-$\mathrel{-\!\!3}_N$.□

Lemma 4.4 *Each rule of* **G3Sn** *is height-preserving invertible.*

Proof For the rules with repetition of the principal formulas in the premiss hp-invertibility follows from Theorem 4.2. For the other rules, if we are inverting w.r.t. the principal formula of the last rule instance in $\mathcal{D}$, there is nothing to prove. Else, we reason by induction on the height of $\mathcal{D}$, possibly applying Lemma 4.1. The base case is trivial, because only atomic formulas are active in initial sequent. To illustrate, assume we are inverting rule $R \dashv_N$ and the last rule instance in $\mathcal{D}$ is the following instance of $R \Vdash$:

$$\frac{Nw, u \in \alpha, \Gamma \Rightarrow \Delta', w : B \dashv C, u : A}{Nw, \Gamma \Rightarrow \Delta', \alpha \Vdash A, w : B \dashv C}\ R\Vdash,\ u \text{ fresh}$$

We transform $\mathcal{D}$ into the following derivation having at most the same height:

$$\cfrac{\cfrac{\cfrac{Nw, u \in \alpha, \Gamma \Rightarrow \Delta', w : B \dashv C, u : A}{Nw, u' \in \alpha, \Gamma \Rightarrow \Delta', w : B \dashv C, u' : A}\ \textit{Lem.4.1}}{Nw, w' : B, wRw', u' \in \alpha, \Gamma \Rightarrow \Delta', u' : A, w' : C}\ IH}{Nw, w' : B, wRw', \Gamma \Rightarrow \Delta', \alpha \Vdash A, w' : C}\ R\Vdash$$

where the substitutions are needed if $w' \equiv u$. □

Theorem 4.5 (Contraction) **G3Sn** $\vdash^n \Pi, \Pi, \Gamma \Rightarrow \Delta, \Sigma, \Sigma$ *implies* **G3Sn** $\vdash^n \Pi, \Gamma \Rightarrow \Delta, \Sigma$.

Proof By induction on the height of the derivation $\mathcal{D}$ of $\Pi, \Pi, \Gamma \Rightarrow \Delta, \Sigma, \Sigma$. If $\Pi, \Pi, \Gamma \Rightarrow \Delta, \Sigma, \Sigma$ is an initial sequent, then the proof is immediate. If the principal formula of last rule applied is not in Π or Σ, then the conclusion follows from an application of the induction hypothesis to the premise(s) and then of the rule. If the principal formula of the last rule applied is in Π or Σ, we exploit the invertibility of the corresponding rule. We give a concrete example of this qualitative analysis.

Let us assume the conclusion of $\mathcal{D}$ is $w : A \dashv B, w : A \dashv B, \Pi', \Pi', \Gamma' \Rightarrow \Delta, \Sigma, \Sigma$. we have two cases depending on whether *Rule* is an instance of $L \dashv_N$ or of $L \dashv_Q$. In the former case we can proceed as when no instance of $w : A \dashv B$ is principal since $L \dashv_N$ is a rule with repetition of the principal formulas. In the latter case we transform

$$\frac{\alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, w : A \dashv B, \Pi', \Pi', \Gamma'' \Rightarrow \Delta, \Sigma, \Sigma}{Qw, w : A \dashv B, w : A \dashv B, \Pi', \Pi', \Gamma'' \Rightarrow \Delta, \Sigma, \Sigma}\ L\dashv_Q,\ \alpha \text{ fresh}$$

into the following derivation of at most the same height:

$$\cfrac{\cfrac{\cfrac{\cfrac{\alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, w : A \dashv B, \Pi', \Pi', \Gamma'' \Rightarrow \Delta, \Sigma, \Sigma}{\beta \in Iw, \beta \Vdash A \supset B, \beta \lhd A \supset B, \alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, \Pi', \Pi', \Gamma'' \Rightarrow \Delta, \Sigma, \Sigma}\ \text{Lem.4.4}}{\alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, \Pi', \Pi', \Gamma'' \Rightarrow \Delta, \Sigma, \Sigma}\ \text{Lem.4.1}}{\alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, \Pi', \Gamma'' \Rightarrow \Delta, \Sigma}\ IH}{Qw, w : A \dashv B, \Pi', \Gamma'' \Rightarrow \Delta, \Sigma}\ L\dashv_Q$$

where both α and β do not occur in the conclusion. □

Theorem 4.6 (Cut) *Let E be a relational, forcing, or covering formula. The following rule of cut is admissible in* **G3Sn***:*

$$\frac{\Gamma \Rightarrow \Delta, E \qquad E, \Pi \Rightarrow \Sigma}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ \textit{Cut}$$

Proof We consider an uppermost instance of *Cut* and we proceed by induction on the degree of the cut-formula with a sub-induction on the cut-height of $\mathcal{D}$—i.e., the sum of the heights of the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of the two premisses. The theorem then follows by induction on the number of cuts in the derivation.

As usual it is convenient to divide the proof in three exhaustive cases: in case (i) one premiss has a derivation of height 1; in case (ii) the cut-formula is not principal in the last step of at least one of the two premisses; in case (iii) the cut-formula is principal in the last step of both premisses.

The proof of cases (i) and (ii) as well as the sub-cases of case (iii) where the principal operator of the cut-formula is in $\wedge, \vee, \rightarrow$, are standard and can thus be omitted. The proof of the sub-cases of (iii) when the cut-formula has shape $\alpha \Vdash A$ or $\alpha \lhd A$ can be found in [11]. Hence, we have to consider only the sub-cases of (iii) where the cut-formula has shape $w : B \mathrel{-\!\!3} C$ and either the multiset Nw, wRv or $Qw, \alpha \in Iw$ occurs in Γ.

In the first case suppose $\mathcal{D}$ is a follows (for u not in the conclusion):

$$\dfrac{\dfrac{\overset{\vdots\mathcal{D}_{11}}{u : B, wRu, Nw, \Gamma' \Rightarrow \Delta, u : C}}{Nw, \Gamma' \Rightarrow \Delta, w : B \mathrel{-\!\!3} C}\,{\scriptstyle R \mathrel{-\!\!3}_N} \qquad \dfrac{\overset{\vdots\mathcal{D}_{21}}{Nw, wRv, w : B \mathrel{-\!\!3} C, \Pi' \Rightarrow \Sigma, v : B} \quad \overset{\vdots\mathcal{D}_{22}}{v : C, Nw, wRv, w : B \mathrel{-\!\!3} C, \Pi' \Rightarrow \Sigma}}{w : B \mathrel{-\!\!3} C, Nw, wRv, \Pi' \Rightarrow \Sigma}\,{\scriptstyle L \mathrel{-\!\!3}_N}}{Nw, Nw, wRv, \Pi', \Gamma' \Rightarrow \Delta', \Sigma}\,{\scriptstyle Cut}$$

We transform it into the following derivation ($[\Gamma]^n$ stands for n copies of Γ, and, for the sake of space, we omit the premisses of dotted inferences):

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\mathcal{D}_1 \quad \vdots\mathcal{D}_{21}}{[Nw]^2, wRv, \Pi', \Gamma' \Rightarrow \Delta, \Sigma, v : B}\,{\scriptstyle Cut_j} \qquad \dfrac{\vdots\mathcal{D}_{11}}{v : B, wRv, Nw, \Gamma' \Rightarrow \Delta, v : C}\,{\scriptstyle Lem.4.1}}{[Nw]^3, [wRv]^2, \Pi', [\Gamma']^2 \Rightarrow [\Delta]^2, \Sigma, v : C}\,{\scriptstyle Cut_i} \qquad \dfrac{\vdots\mathcal{D}_1 \quad \vdots\mathcal{D}_{22}}{v : C, [Nw]^2, wRv, \Pi', \Gamma' \Rightarrow \Delta, \Sigma}\,{\scriptstyle Cut_j}}{[Nw]^5, [wRv]^3, [\Pi']^2, [\Gamma']^3 \Rightarrow [\Delta]^3, [\Sigma]^2}\,{\scriptstyle Cut_i}}{Nw, wRv, \Pi, \Gamma \Rightarrow \Delta, \Sigma}\,{\scriptstyle Lem.4.5}$$

where instances of *Cut* with subscript i (j) are admissible by the (sub-)induction hypothesis.

Finally, if $\mathcal{D}$ is a follows (for β not in the conclusion):

$$\dfrac{\dfrac{\overset{\vdots\mathcal{D}_{11}}{Qw, \beta \in Iw, \Gamma' \Rightarrow \Delta, w : B \mathrel{-\!\!3} C, \beta \Vdash B \supset C} \quad \overset{\vdots\mathcal{D}_{12}}{Qw, \beta \in Iw, \Gamma' \Rightarrow \Delta, w : B \mathrel{-\!\!3} C, \beta \lhd B \supset C}}{Qw, \beta \in Iw, \Gamma' \Rightarrow \Delta, w : B \mathrel{-\!\!3} C}\,{\scriptstyle R \mathrel{-\!\!3}_Q} \qquad \dfrac{\overset{\vdots\mathcal{D}_{21}}{\alpha \in Iw, \alpha \Vdash B \supset C, \alpha \lhd B \supset C, Qw, \Pi' \Rightarrow \Sigma}}{Qw, w : B \mathrel{-\!\!3} C, \Pi' \Rightarrow \Sigma}\,{\scriptstyle L \mathrel{-\!\!3}_Q}}{Qw, \alpha \in Iw, \Pi', \Gamma' \Rightarrow \Delta, \Sigma}\,{\scriptstyle Cut}$$

we transform it into the following derivation ($\mathcal{D}_{1i}[\star]$ stands for the derivation $\mathcal{D}_{1i}$ with α in place of β by an instance of Lemma 4.1, and D stands for $B \supset C$):

$$\dfrac{\dfrac{\dfrac{\vdots\mathcal{D}_{12}[\star] \quad \vdots\mathcal{D}_2}{[Qw]^2, \alpha \in Iw, \Pi', \Gamma' \Rightarrow \Delta, \Sigma, \alpha \lhd D}\,{\scriptstyle Cut_j} \qquad \dfrac{\dfrac{\vdots\mathcal{D}_{11}[\star] \quad \vdots\mathcal{D}_2}{[Qw]^2, \alpha \in Iw, \Pi', \Gamma' \Rightarrow \Delta, \Sigma, \alpha \Vdash D}\,{\scriptstyle Cut_j} \quad \overset{\vdots\mathcal{D}_{21}}{\alpha \Vdash D, \alpha \lhd D, \alpha \in Iw, Qw, \Pi' \Rightarrow \Sigma}}{\alpha \lhd D, [Qw]^3, [\alpha \in Iw]^2, [\Pi']^2, \Gamma' \Rightarrow \Delta, [\Sigma]^2}\,{\scriptstyle Cut_i}}{[Qw]^5, [\alpha \in Iw]^3, [\Pi']^3, [\Gamma']^2 \Rightarrow [\Delta]^2, [\Sigma]^3}\,{\scriptstyle Cut_i}}{Nw, \alpha \in Iw, \Pi, \Gamma \Rightarrow \Delta, \Sigma}\,{\scriptstyle Thm.4.5}$$

□

Corollary 4.7 *The rule* $MP_{\mathrel{-\!\!3}}$ *is* **G3Sn**-*admissible:*

$$\dfrac{Nw \Rightarrow w : A \mathrel{-\!\!3} B \qquad Nw \Rightarrow w : A}{Nw \Rightarrow w : B}\,{\scriptstyle Det.}$$

Proof By applying Lemma 4.4 to $Nw \Rightarrow w : A \mathrel{-\!\!3} B$ we obtain the derivability of $u : A, wRu, Nw \Rightarrow u : B$ for some fresh label u. By an instance of Lemma 4.1 this becomes $w : A, wRw, Nw \Rightarrow w : B$ and, by a *Cut* with $Nw \Rightarrow w : A$, we obtain $wRw, Nw, Nw \Rightarrow w : B$. Finally, we apply an instance of Rule Ref_R and one of Theorem 4.5 to conclude that $Nw \Rightarrow w : B$ is derivable. □

Corollary 4.8 **G3Sn**-*derivations are analytic, i.e. every label occurring in a derivation either occurs in its conclusion or it is an* eigenvariable *and every formula is a subformula of the formulas in the conclusion.*

Proof See [14, Lemma 3.17]. □

Lemma 4.9 *For every formula A, B and C, if the sequents $w : A \Rightarrow w : B$ and $w : B \Rightarrow w : A$ are derivable in* **G3Sn**, *then the sequents:*

$$w : C \Rightarrow w : C[A//B] \text{ and } w : C[A//B] \Rightarrow w : C$$

are provable in **G3Sn**.

Proof The proof runs by induction on the weight of the formula C. We assume that $C \not\equiv A$, otherwise the proof is trivial. If C is a sentential variable p or $\bot$, the claim is trivial. If C is a conjunction, a disjunction or a formula of the shape $D \supset E$, then the proof easily follows by applying the induction hypothesis. We discuss the case in which C is of the form $D \mathrel{-\!\!3} E$. Since $C \not\equiv A$, we have $(D \mathrel{-\!\!3} E)[A//B] \equiv D[A//B] \mathrel{-\!\!3} E[A//B]$.

We first show that $Nw, w : D \mathrel{-\!\!3} E \Rightarrow w : D[A//B] \mathrel{-\!\!3} E[A//B]$ is derivable.

$$\dfrac{\dfrac{[...], w : D \mathrel{-\!\!3} E, u : D[A//B] \Rightarrow u : E[A//B], u : D \qquad [...], w : D \mathrel{-\!\!3} E, u : D[A//B], u : E \Rightarrow u : E[A//B]}{Nw, wRu, w : D \mathrel{-\!\!3} E, u : D[A//B] \Rightarrow u : E[A//B]}\text{L}\mathrel{-\!\!3}_N}{Nw, w : D \mathrel{-\!\!3} E \Rightarrow w : D[A//B] \mathrel{-\!\!3} E[A//B]}\text{R}\mathrel{-\!\!3}_N$$

The derivability of the topmost sequents follows from the induction hypothesis and weakening. The sequent $Qw, w : D \mathrel{-\!\!3} E \Rightarrow w : D[A//B] \mathrel{-\!\!3} E[A//B]$ is derivable too.

$$\dfrac{\dfrac{\dfrac{\dfrac{[...], o \in \alpha, o : D \supset E \Rightarrow o : D[A//B] \supset E[A//B]}{[...], o \in \alpha, \alpha \Vdash D \supset E \Rightarrow o : D[A//B] \supset E[A//B]}\text{L}\Vdash}{[...], \alpha \Vdash D \supset E \Rightarrow \alpha \Vdash D[A//B] \supset E[A//B]}\text{R}\Vdash \qquad \dfrac{\dfrac{[...], u : D[A//B] \supset E[A//B], \alpha \lhd D \supset E \Rightarrow u : D \supset E}{[...], u \notin \alpha, u : D[A//B] \supset E[A//B], \alpha \lhd D \supset E \Rightarrow}\text{L}\lhd}{[...], \alpha \lhd D \supset E \Rightarrow \alpha \lhd D[A//B] \supset E[A//B]}\text{R}\lhd}{Qw, \alpha \in Iw, \alpha \lhd D \supset E, \alpha \Vdash D \supset E \Rightarrow w : D[A//B] \mathrel{-\!\!3} E[A//B]}\text{R}\mathrel{-\!\!3}_Q}{Qw, w : D \mathrel{-\!\!3} E \Rightarrow w : D[A//B] \mathrel{-\!\!3} E[A//B]}\text{L}\mathrel{-\!\!3}_Q$$

The derivability of the topmost sequents follows from application of the rules L$\supset$, R$\supset$, the induction hypothesis and weakening. The desired conclusion follows from an application of $Norm$.

We now discuss the other part of the claim, i.e. $w : D[A//B] \mathrel{-\!\!3} E[A//B] \Rightarrow w : D \mathrel{-\!\!3} E$. We first show that $Nw, w : D[A//B] \mathrel{-\!\!3} E[A//B] \Rightarrow w : D \mathrel{-\!\!3} E$:

$$\dfrac{\dfrac{[...], u : D \Rightarrow u : D[A//B], u : E \qquad [...],, u : E[A//B], u : D \Rightarrow u : E}{[...], w : D[A//B] \mathrel{-\!\!3} E[A//B], u : D \Rightarrow u : E}\text{L}\mathrel{-\!\!3}_N}{Nw, w : D[A//B] \mathrel{-\!\!3} E[A//B] \Rightarrow w : D \mathrel{-\!\!3} E}\text{R}\mathrel{-\!\!3}_N$$

Again, the derivability of the topmost sequents follows from the induction hypothesis and weakening. For the other direction we proceed as follows (we omit to display redundant repetition of formulas):

$$\dfrac{\dfrac{\dfrac{[...], o : D[A//B] \supset E[A//B] \Rightarrow o : D \supset E, [...]}{[...], o \in \alpha, \alpha \Vdash D[A//B] \supset E[A//B] \Rightarrow \alpha \Vdash D \supset E, [...]}\text{L}\Vdash \qquad \dfrac{\dfrac{[...], u : D \supset E \Rightarrow u : D[A//B] \supset E[A//B], [...]}{[...], u \notin \alpha, u : D \supset E, \alpha \lhd D[A//B] \supset E[A//B] \Rightarrow [...]}\text{L}\lhd}{[...], \alpha \lhd D[A//B] \supset E[A//B] \Rightarrow \alpha \lhd D \supset E, [...]}\text{R}\lhd}{Qw, \alpha \in Iw, \alpha \Vdash D[A//B] \supset E[A//B], \alpha \lhd D[A//B] \supset E[A//B] \Rightarrow w : D \mathrel{-\!\!3} E}\text{R}\mathrel{-\!\!3}_Q}{Qw, w : D[A//B] \mathrel{-\!\!3} E[A//B] \Rightarrow w : D \mathrel{-\!\!3} E}\text{L}\mathrel{-\!\!3}_Q$$

The topmost sequents are derivable via applications of the rules R$\supset$, L$\supset$, the induction hypothesis and admissibility of weakening.

□

We shall now prove the admissibility of the rule of substitution of strict equivalents.

Corollary 4.10 *The rule of substitution of strict equivalents is* **G3Sn**-*admissible:*

$$\frac{Nw \Rightarrow w : A \qquad Nw \Rightarrow w : (B \prec C) \wedge (C \prec B)}{Nw \Rightarrow w : A[B//C]}\ SSE$$

Proof We assume that we have a proof of the sequents $Nw \Rightarrow w : A$ and $Nw \Rightarrow w : (B \prec C) \wedge (C \prec B)$. By invertibility of the rule R$\wedge$ we get the derivations of $Nw \Rightarrow w : B \prec C$ and $Nw \Rightarrow w : C \prec B$.

We apply again the invertibility of the rule R$\prec$ we get $Nw, wRu, u : B \Rightarrow u : C$ and $Nw, wRu, u : C \Rightarrow u : B$. By inspection of the rules, the normality atoms and the relational atoms displayed in such sequents are never active in a derivation, so we can remove them.

Therefore the sequents $u : C \Rightarrow u : B$ and $u : B \Rightarrow u : C$ are derivable and we can apply Lemma 4.9 which yields $w : A \Rightarrow w : A[B//C]$. Finally, a cut gives the desired result. □

We are now in the position to state and prove the embedding of the axiomatic calculi **Sn** into **G3Sn**.

Theorem 4.11 *If* **Sn** $\vdash A$*, then* **G3Sn** $\vdash Nw \Rightarrow w : A$.

Proof The proof runs by induction on the height of the derivation in the axiomatic calculi **Sn**. The axioms are derivable by Lemma 4.3. The rule *Adj* is admissible by rule R$\wedge$. The admissibility of $MP_{\prec}$ is a consequence of the Corollary 4.7, and that of *SSE* follows from Theorem 4.10. □

5 Characterisation

We will now propose an alternative and more direct form of completeness which is obtained by extracting a countermodel out of a failed proof search. We start by defining the notion of validity of labelled sequents [11].

Definition 5.1 Given a set $\mathbb{W}'$ of world labels w, a set $\mathbb{I}'$ of neighborhood labels α and an **Sn** model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{R}, \mathcal{I}, \mathcal{V} \rangle$, an SN realisation (ρ, σ) is a pair of functions mapping each $w \in \mathbb{W}'$ into $\rho(x) \in \mathcal{W}$ and mapping each $\alpha \in \mathbb{I}'$ into $\sigma(\alpha) \in \mathcal{I}w$ for some $w \in \mathcal{W}$. We introduce the notion $\mathcal{M}$ *satisfies a formula E under an $\mathbb{W}'\mathbb{I}'$-realisation* (σ, ρ) and denote it by $\mathcal{M} \vDash_{\rho,\sigma} E$, where we assume that the labels in E occur in $\mathbb{W}', \mathbb{I}'$. The definition extends by cases on the form of E, we give some examples:

- $\mathcal{M} \vDash_{\rho,\sigma} w \in \alpha$ if $\rho(w) \in \sigma(\alpha)$.
- $\mathcal{M} \vDash_{\rho,\sigma} w : A$ if $\vDash_{\rho(w)} A$
- $\mathcal{M} \vDash_{\rho,\sigma} \alpha \Vdash A$ if for all u in $\sigma(\alpha)$, $\rho(u) \vDash A$

- $\mathcal{M} \vDash_{\rho,\sigma} \alpha \lhd A$ if for all u s. t. $\mathcal{M} \vDash_{\rho,\sigma} u : A$, $\rho(u) \in \sigma(\alpha)$.

Given a sequent $\Gamma \Rightarrow \Delta$, let $\mathbb{W}', \mathbb{I}'$ be the sets of world and neighborhood labels occurring in $\Gamma \cup \Delta$, and let (ρ, σ) be an $\mathbb{W}', \mathbb{I}'$-realisation; we define $\mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta$ to hold if, whenever $\mathcal{M} \vDash_{\rho,\sigma} E$ for all formulas $E \in \Gamma$, then $\mathcal{M} \vDash_{\rho,\sigma} F$ for some formula $F \in \Delta$. We further define $\mathcal{M}$-validity by:

$$\mathcal{M} \vDash \Gamma \Rightarrow \Delta \text{ iff } \mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta \text{ for every } SN\text{-realisation } (\rho, \sigma).$$

A sequent $\Gamma \Rightarrow \Delta$ is **Sn**-*valid* if $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$ for every **Sn**-model $\mathcal{M}$.

Theorem 5.2 (Soundness) *If* **G3Sn** $\vdash \Gamma \Rightarrow \Delta$*, then* $\Gamma \Rightarrow \Delta$ *is* **Sn**-*valid.*

Proof By induction on the height of the derivations in the calculus **G3Sn**. □

We introduce the notion of saturated sequent in a derivation. For every branch in a derivation we write $\downarrow \Gamma$ ($\downarrow \Delta$) to denote the union of the antecedents (succedents) in the branch from the endsequent up to the sequent $\Gamma \Rightarrow \Delta$.

Definition 5.3 A branch in a proof search in the system **G3S1** from the endsequent up to the sequent $\Gamma \Rightarrow \Delta$ is *saturated* if, for every rule R, if the principal formulas of R occur in the branch, the formulas introduced by one of the premises of R also occur in the branch. In detail, a saturated branch up to $\Gamma \Rightarrow \Delta$ has to satisfy the following conditions (we omit some of them): (Ax) There is no sentential variable p such that $w : p \in \Gamma \cap \Delta$. ($\text{Ax}_C$) There are no α, w such that $w \in \alpha, w \notin \alpha, \in \Gamma$. ($\text{Ax}_N$) There is no w such that $Nw, Qw, \in \Gamma$. (L$\bot$) It is not the case that $w : \bot \in \Gamma$. (L$\multimap_Q$) If Qw and $w : A \multimap B \in\downarrow \Gamma$, then $\alpha \in Iw$, $\alpha \lhd A \supset B$ and $\alpha \Vdash A \supset B$ are in $\downarrow \Gamma$ for some α. (R$\multimap_Q$) If Qw, $\alpha \in Iw$ are in $\downarrow \Gamma$ and $w : A \multimap B \in\downarrow \Delta$, then $\alpha \lhd A \supset B \in\downarrow \Delta$ or $\alpha \Vdash A \supset B \in\downarrow \Delta$. The notion of saturated sequent is extended to the systems **G3Sn** by adding conditions relative to the additional rules.

Given a sequent $\Gamma \Rightarrow \Delta$ we build a *proof search tree* by applying all possible rules of the calculus. To avoid repetitions, we fix a counter. At stage 1 we apply rule L$\wedge$, at stage 2 the rule R$\wedge$ and so forth. There are 18 + m different stages (where m is the number of relational and additional rules depending on the system). At stage 18 + m + 1 we repeat stage 1. If the construction ends we obtain a derivation or a finite tree in which a branch is saturated, otherwise we obtain an infinite tree. By König's Lemma there is an infinite branch which is saturated from which we can extract a countermodel.

Theorem 5.4 *Given a saturated branch* $\mathcal{B}$ *in a proof search tree for a sequent* $\Pi \Rightarrow \Sigma$ *up to the sequent* $\Gamma \Rightarrow \Delta$ *built according to the rules of system* **G3Sn***, we can extract a countermodel* $\mathcal{M}$ *for the endsequent based on an* **Sn**-*frame.*

Proof Given a saturated branch $\mathcal{B}$ up to $\Gamma \Rightarrow \Delta$ in a proof search tree for the endsequent $\Pi \Rightarrow \Sigma$ we define the following countermodel: $\langle \mathcal{W}, \mathcal{N}, \mathcal{R}, \mathcal{I}, \mathcal{V} \rangle$ such that:

- $\mathcal{W}$ is the set of all world labels occurring in $\downarrow \Gamma$.
- $\mathcal{N}$ is the set of all labels w such that $Nw \in\downarrow \Gamma$.
- $w\mathcal{R}u$ if and only if wRu occurs in $\downarrow \Gamma$.

- $\mathcal{I}(w)$ is the set of all the neighbors α such that $\alpha \in Iw$ occurs in $\downarrow \Gamma$ and every α consists of all the worlds w such that $w \in \alpha$ occur in $\downarrow \Gamma$.
- $\mathcal{V}(p)$ is the set of all worlds w such that $w : p$ occurs in $\downarrow \Gamma$.

Notice that $\mathcal{V}$ is well defined by the saturation conditions Ax, Ax_C, Ax_N, and $L\bot$. For every system **G3Sn**, the frame $\langle \mathcal{W}, \mathcal{N}, \mathcal{R}, \mathcal{I} \rangle$ satisfies the properties of **Sn**-frames by the saturation conditions regarding relational and additional rules. We define the realization (ρ, σ) such that $\rho(w) \equiv w$ and $\sigma(\alpha) \equiv \alpha$. We claim that:

(i) If $w : A$ is in $\downarrow \Gamma$, then $\mathcal{M} \vDash_{\rho,\sigma} w : A$.

(ii) If $w : A$ is in $\downarrow \Delta$, then $\mathcal{M} \nvDash_{\rho,\sigma} w : A$.

The proof is by simultaneous induction on the degree of A. We focus on the case of strict implication.

(a) If $w : A \multimap B$ is in $\downarrow \Gamma$, then by the saturation condition there is either Qw or Nw in $\downarrow \Gamma$. In the first case, again by the saturation condition, there are $\alpha \in Iw$, $\alpha \lhd A \supset B$ and $\alpha \Vdash A \supset B$ in $\downarrow \Gamma$. By definition of $\mathcal{M}$ and induction hypothesis we have $\alpha \in \mathcal{I}(w)$, $\mathcal{M} \vDash_{\rho,\sigma} \alpha \lhd A \supset B$ and $\mathcal{M} \vDash_{\rho,\sigma} \alpha \Vdash A \supset B$, therefore $\mathcal{M} \vDash_{\rho,\sigma} w : A \multimap B$. In the second case, we distinguish two subcases. If there is no label u such that wRu occurs in $\downarrow \Gamma$, then the claim trivially follows. Otherwise for every u sucht that wRu occurs in $\downarrow \Gamma$, by the saturation condition either $u : A$ is in $\downarrow \Delta$ or $u : B$ is in $\downarrow \Gamma$. By induction hypothesis we get $\mathcal{M} \nvDash_{\rho,\sigma} u : A$ or $\mathcal{M} \vDash_{\rho,\sigma} u : B$. Therefore we get $\mathcal{M} \vDash_{\rho,\sigma} w : A \multimap B$.

(b) If $w : A \multimap B$ is in $\downarrow \Delta$, then by the saturation condition there is either Qw or Nw in $\downarrow \Gamma$. In the first case, by the saturation condition, for every $\alpha \in \mathcal{I}(w)$, there is $\alpha \lhd A \supset B$ or $\alpha \Vdash A \supset B$ in $\downarrow \Delta$. In both cases by induction hypothesis it follows $\mathcal{M} \nvDash_{\rho,\sigma} w : A \multimap B$. In the second case, by saturation there are $wRu, u : A \in\downarrow \Gamma$ and $u : B$ in $\downarrow \Delta$. By induction hypothesis we get $\mathcal{M} \vDash_{\rho,\sigma} u : A$ and $\mathcal{M} \nvDash_{\rho,\sigma} u : B$, which yields $\mathcal{M} \vDash_{\rho,\sigma} w : A \multimap B$.

□

Corollary 5.5 (Completeness) *For every formula A:*

$$\mathbf{Sn} \vDash A \text{ if and only if } \mathbf{G3Sn} \vdash Nw \Rightarrow w : A$$

Proof The direction from right to left is the content of the soundness theorem. For the other direction we prove the contrapositive. Suppose that **G3Sn** $\nvdash Nw \Rightarrow w : A$, hence there is a saturated branch and we can extract a **Sn**-countermodel for $Nw \Rightarrow w : A$, which gives **Sn** $\nvDash A$. □

We observe that our completeness proof builds a natural bridge from the neighborhood semantics for **S1** to a bi-neighborhood one, see [4]. Indeed, to build the countermodel out of a failed proof search we are actually considering the complement of every neighbor α, but the rules of our calculus naturally build a pair of disjoint sets for every modal operator as in the case of bi-neighborhood semantics.

6 Syntactic decidability of S1

In this section we use the labelled system **G3S1** to obtain a syntactic decidability result for **S1** via terminating proof search. The decidability proofs in the literature for **S1** are obtained via semantic methods showing that the system satisfies the finite model property [2].

To establish decidability we need to show that the search for a derivation can be interrupted at a certain point and that we can extract a finite countermodel if the search fails. Termination of the proof search and completeness entail decidability. However, the extraction of a countermodel can be regarded as a *desideratum* as it yields a constructive proof of the finite model property. First we prove some preliminary lemmata.

Lemma 6.1 *The rules Ref_R, Ref_I, $S1$, $Norm$, $L\Vdash$, $L\lhd$, $L\text{-}3_N$ and $R\text{-}3_Q$ need not be instantiated more than once on the same principal formula(s) in every branch in a proof search.*

Proof By height-preserving admissibility of contraction. □

In order to obtain a termination result we need to show that the number of labels introduced in a proof search is finite. Therefore we need to establish some bounds on the application of the *dynamic* rules—i.e., the rules which introduce new labels, which are R$\Vdash$, $S1$, R$\lhd$, L$\text{-}3_Q$ and R$\text{-}3_N$. We now introduce some definitions which allow us to check the relations between world labels and neighborhood labels.

Definition 6.2 In a branch $\mathcal{B}$ of a proof search tree of the sequent $Nw \Rightarrow w : A$ we define the relation $\rightarrow_{\mathcal{B}}$ of *immediate successor* (for $u, v \in \mathbb{W}$ and $\alpha \in \mathbb{I}$): (i) $u \rightarrow_{\mathcal{B}} \alpha$ if $\alpha \in Iu$ occurs in $\mathcal{B}$; (ii) $\alpha \rightarrow_{\mathcal{B}} u$ if in $\mathcal{B}$ there is either $u \in \alpha$ or $u \notin \alpha$; (iii) $u \rightarrow_{\mathcal{B}} v$ uRv is in $\mathcal{B}$.

Fact. 1 The transitive closure of $\rightarrow_{\mathcal{B}}$ defines a tree which, as it is easy to check, does not contain cycles *modulo* the reflexive ones.
Fact. 2 The immediate successors of a world label in a saturated brach of a proof search tree are either all neighborhood labels or world labels, but not both.

Theorem 6.3 *Each label in a branch $\mathcal{B}$ of a proof search tree of an endsequent $Nw \Rightarrow w : A$ has only a finite number of immediate successors.*

Proof The immediate successors of a label can be introduced only by applications of the dynamic rules R$\Vdash$, $S1$, R$\lhd$, L$\text{-}3_Q$ or R$\text{-}3_N$. The subformulas of the formula A which occur in a proof search are finite by Corollary 4.8, therefore if there were infinite immediate successors there would be more than one application of one of the above mentioned rules to the same principal labelled formulas. We show that every derivation can be transformed in a derivation of the same height in which every branch contains at most one application of such rules to the same principal labelled formulas. We detail the case of L$\text{-}3_Q$ as an example; notice that by Fact 2 we can assume that the formula is again principal in an application of L$\text{-}3_Q$ and not of L$\text{-}3_N$.

$$\frac{Qu, \beta \in Iu, \beta \Vdash A \supset B, \beta \lhd A \supset B, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}{Qu, u : A \dashv B, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}\ \text{L-}\dashv_Q$$

$$\vdots\ \mathcal{D}$$

$$\frac{Qu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}{Qu, u : A \dashv B, \Gamma \Rightarrow \Delta}\ \text{L-}\dashv_Q$$

We transform the derivation as follows:

$$\cfrac{\cfrac{\cfrac{Qu, \beta \in Iu, \beta \Vdash A \supset B, \beta \lhd A \supset B, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}{Qu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}\ \text{Lem.4.1}}{Qu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}\ \text{Thm.4.5}}{Qu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, u : A \dashv B, \Gamma \Rightarrow \Delta}\ \text{Thm.4.2}$$

$$\vdots\ \mathcal{D}$$

$$\frac{Qu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}{Qu, u : A \dashv B, \Gamma \Rightarrow \Delta}\ \text{L-}\dashv_Q$$

The application of the hp-admissible rules of substitution, contraction and weakening does not introduce new applications of L-$\dashv$ (this is easily checked). □

As a consequence, the tree defined by $\rightarrow_{\mathcal{B}}$ is finitely branching. The second part of the proof of termination consists in showing that in every branch the length of a chain of labels is finite. This depends on the fact that the relation defined by $\rightarrow_{\mathcal{B}}$ in a proof search is intransitive. In particular, a label *sees* only its immediate successors and itself (by reflexivity). Therefore the length of a branch is determined by the number of modal operators occurring in a formula.

Theorem 6.4 *Every chain of labels in a branch in a proof search for the sequent $Nw \Rightarrow w : A$ is finite.*

Proof Given a chain of labels in a branch in a proof search for the sequent $Nw \Rightarrow w : A$ and a label u in the chain, every immediate successor of u is introduced by the application of one of the dynamical rules R-$\dashv_N$, L-$\dashv_Q$, R$\lhd$, R$\Vdash$ or $S1$ to a formula B labelled by u. By inspection, these rules can be applied whenever B contains at least one modal operator. However, since every label sees only itself and its immediate successors, every label introduced by the analysis of $u : B$ will label only formulas of lesser degree. Since by definition the degree of each formula is finite, the chain is finite. □

Theorem 6.5 *The proof search for a sequent $Nw \Rightarrow w : A$ in the system* **G3S1** *terminates.*

Proof The proof is immediate because in every branch the number of labels generated is finite. □

Corollary 6.6 *The relation* **G3S1** $\vdash Nw \Rightarrow w : A$ *is decidable.*

Proof By Theorem 5.4 we can extract a countermodel out of a saturated branch, so we get the finite model property and the decidability of the system. □

7 Conclusion

We introduced a modular and uniform approach to the proof theory of the strict implication logics by C. I. Lewis. By converting the truth conditions of the semantics into suitably formulated rules of the calculus, we obtained labelled systems with good structural properties. Furthermore, the analyticity of the systems enabled us to obtain a syntactic proof of the decidability of the system **S1**. We conjecture that the upper bound given by the proof search procedure is not optimal, but we have not investigated this and other related complexity issues yet.

There are some possible themes for future research. First, it would be interesting to see whether it is possible to obtain calculi for systems related to **S1**, cf. [2]. Second, the semantics of **S1**, or a modification theoreof, might be employed to model hyperintensional features—see [1,5,12]—in the context of epistemic logics. Finally, it might be interesting to see whether the semantics for **S1** can be simplified and if it can be adapted to the bi-neighborhood framework.

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