Decidable first-order modal logics with counting quantifiers

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Abstract

In this paper, we examine the computational complexity of various natural one-variable fragments of first-order modal logics with the addition of arbitrary counting quantifiers. The addition of counting quantifiers provides us a rich language with which to succinctly express statements about the quantity of objects satisfying a given property, using only a single variable. We provide optimal upper bounds on the complexity of the decision problem for several one-variable fragments, by establishing the finite model property. In particular, we show that the decision (validity) problem for the one-variable fragment of the minimal first-order modal logic **QK** with counting quantifiers is coNExpTime-complete. In the propositional setting, these results also provide optimal upper bounds for many two-dimensional modal logics in which one component is von Wright's logic of 'elsewhere'.

Keywords: first-order modal logic, quantified modal logic, two-dimensional modal logic, counting quantifiers, decidable fragment, finite model property, quasimodel

1 Introduction

First-order modal logics are notorious for their poor computational behaviour, and even the modal versions of many decidable fragments of classical first-order logic are undecidable. For example, even the two-variable, monadic fragment of many first-order modal logics is already undecidable [8]. However, more restrictive decidable fragments are known to exist, such as the *monodic* fragment, in which modalities *de re* are restricted to formulas containing at most one free variable, with no restrictions are placed on modalities *de dicto* [22].

In classical first-order logic, counting quantifiers allow us to succinctly express statements about the quantity of objects satisfying a given property, without requiring many auxiliary variables to address each object. It is well-known that counting quantifiers can be safely added to the two-variable fragment of classical first-order logic without affecting the computational complexity [13].

They, therefore, provide an attractive addition in the quest to gain greater expressive power from finite variable fragments of first-order modal logics, without jeopardizing their decidability. Some examples of first-order formulas with counting quantifiers include:

- *"It is possible that there are more than eight planets"*: $\Diamond\exists_{>8}x\ \mathsf{Planet}(x)$
- *"Two components are believed to be faulty"*: $\exists_{=2}x\big(\mathsf{Component}(x) \wedge \Box\mathsf{Faulty}(x)\big)$
- The generalised Barcan formula: $\exists_{=c}x\Diamond P(x) \leftrightarrow \Diamond\exists_{=c}x\ P(x)$, for $c < \omega$,

Unfortunately, many decidable first-order *temporal* logics become undecidable with the addition of counting quantifiers; even if we restrict the quantifiers to those of the form $\exists_{\leq c}x$, for $c = 0, 1$ [7]. On the other hand, less expressive first-order modal logics, such as the one-variable fragment of quantified **S5** with counting quantifiers can be easily embedded into the two-variable fragment of classical first-order logic with counting quantifiers, and are therefore no more complex than their counting-free counterparts. It is, therefore, interesting to establish where the boundary lies between decidable and undecidable fragments of first-order modal logics with counting quantifiers.

In this paper, we provide optimal upper bounds for various one-variable fragments of the quantified versions **K**, **KT**, **KB**, **S5**, and **Alt**, with arbitrary counting quantifiers whose subscripts are encoded as binary strings.

In Section 2, we introduce the definitions for the fragments of first-order modal logics that will be working with, and in Section 3 we prove the main results of this paper. Section 4 describes the connection between certain fragments of first-order modal logics with counting quantifiers and two-dimensional propositional modal logics in which one component is von Wright's logic of 'elsewhere'. We conclude with a discussion of some open problems in Section 5. Supplementary proofs and polynomial reductions between several first-order modal logics are provided in the Appendices.

2 First-order Modal Logics

Given a countably infinite set of predicate symbols $\mathsf{Pred} = \{P_0, P_1, \dots\}$, each with an associated arity, and a countable set of first-order variables Var. Let $\mathcal{Q}^{\#}\mathcal{ML}$ denote the set of all first-order modal formulas with *counting quantifiers* defined by the following grammar:

$$\varphi \ ::= \ P_i(x_1, \dots, x_n) \ \mid \ \neg\varphi \ \mid \ (\varphi_1 \wedge \varphi_2) \ \mid \ \Diamond\varphi \ \mid \ (\exists_{\leq c}x\ \varphi)$$

where $P_i \in \mathsf{Pred}$ is an n-ary predicate symbol, $x, x_1, \dots, x_n \in \mathsf{Var}$ are first-order variables, and $c < \omega$ is a natural number quantifier subscript. Other boolean connectives are defined in the usual way, with the addition of $\forall x\ \varphi := \exists_{\leq 0}x\ \neg\varphi$, $\exists_{\geq c}x\ \varphi := \neg\exists_{\leq(c-1)}x\ \varphi$, and $\exists_{=c}x\ \varphi := \exists_{\geq c}x\ \varphi \wedge \exists_{\leq c}x\ \varphi$, for $c > 0$. The results contained herein can be easily modified to accommodate taking either $\exists_{=c}$ or $\exists_{\geq c}$ as primitive.

Throughout this paper, we will assume that formulas of $\mathcal{Q}^{\#}\mathcal{ML}$ can be encoded as strings over some finite alphabet, and define the *size* of a formula $\varphi \in \mathcal{Q}^{\#}\mathcal{ML}$, denoted $||\varphi||$, to be the length of the encoding string, with quantifier subscripts encoded in *binary.* We define $\mathrm{sub}(\varphi) \subseteq \mathcal{Q}^{\#}\mathcal{ML}$ to be the set of all *subformulas* of φ, $\mathsf{md}(\varphi) < \omega$ to be the *modal depth* of φ, taken to be the maximum nesting depth of modal operators, and $\mathsf{count}(\varphi) < \omega$ to be the value of the largest quantifier subscript occurring in φ.

For each $\ell < \omega$, let $\mathcal{Q}^{\#}\mathcal{ML}^{\ell}$ denote the ℓ-variable fragment comprising only those formulas containing the variables $x_1, \dots, x_\ell$, and denote by $\mathcal{Q}^{\#}\mathcal{ML}_k$ the set of all $\mathcal{Q}^{\#}\mathcal{ML}$ formulas that do not contain quantifiers with subscripts larger than $k < \omega$. In particular, we identify $\mathcal{Q}^{\#}\mathcal{ML}_0$ with the language of regular (counting-free) first-order modal logic. We write $\mathcal{Q}^{\#}\mathcal{ML}_k^{\ell} = \mathcal{Q}^{\#}\mathcal{ML}^{\ell} \cap \mathcal{Q}^{\#}\mathcal{ML}_k$ for the ℓ-variable fragment with quantifiers subscripts not exceeding k.

Formulas of $\mathcal{Q}^{\#}\mathcal{ML}$ are interpreted in *first-order Kripke models* of the form $\mathfrak{M} = (\mathfrak{F}, D, I)$, where $\mathfrak{F} = (W, R)$ is a *Kripke frame*, D is a non-empty *domain*, and I is a function associating each $w \in W$ with a first-order structure

$$I(w) = \left\langle D, P_0^{I(w)}, P_1^{I(w)}, \dots \right\rangle$$

where $P_i^{I(w)} \subseteq D^n$ is an n-ary relation on the domain D, for every n-ary predicate symbol $P_i \in \mathsf{Pred}$. The *size of* $\mathfrak{M}$ is taken to be $|W| \cdot |D|$.

In this paper we consider only logics that can be characterised by models with constant domains. However, all the results proved here can be extended to those cases characterised by models with *expanding* or *decreasing* domains via a standard reduction [22].

A *variable assignment* on $\mathfrak{M}$ is a function $\mathfrak{a} : \mathsf{Var} \to D$ mapping variables to elements of the domain. Given a model $\mathfrak{M} = (\mathfrak{F}, D, I)$ and a variable assignment $\mathfrak{a}$, we define satisfiability in $\mathfrak{M}$ by taking, for all $w \in W$:

$$\begin{aligned}
\mathfrak{M}, w \models^{\mathfrak{a}} P_i(x_1, \dots, x_n) \quad &\iff \quad \big(\mathfrak{a}(x_1), \dots, \mathfrak{a}(x_n)\big) \in P_i^{I(w)}, \\
\mathfrak{M}, w \models^{\mathfrak{a}} \neg\varphi \quad &\iff \quad \mathfrak{M}, w \not\models^{\mathfrak{a}} \varphi, \\
\mathfrak{M}, w \models^{\mathfrak{a}} (\varphi_1 \wedge \varphi_2) \quad &\iff \quad \mathfrak{M}, w \models^{\mathfrak{a}} \varphi_1 \text{ and } \mathfrak{M}, w \models^{\mathfrak{a}} \varphi_2, \\
\mathfrak{M}, w \models^{\mathfrak{a}} \Diamond\varphi \quad &\iff \quad wRv \text{ and } \mathfrak{M}, v \models^{\mathfrak{a}} \varphi, \text{ for some } v \in W,
\end{aligned}$$

where $P_i \in \mathsf{Pred}$ is an n-ary predicate symbol, and

$$\mathfrak{M}, w \models^{\mathfrak{a}} (\exists_{\leq c} x\ \varphi) \quad \iff \quad \Big| \{a \in D \ : \ \mathfrak{M}, w \models^{\mathfrak{a}(x/a)} \varphi\} \Big| \ \leq \ c,$$

for $c < \omega$, where $|X|$ denotes the cardinality of X, and $\mathfrak{a}(x/a) : \mathsf{Var} \to D$ is the variable assignment that agrees with $\mathfrak{a}$ on all variables except x, for which it assigns the value $a \in D$.

A formula φ is said to be *satisfiable* in a model $\mathfrak{M} = (\mathfrak{F}, D, I)$, based on $\mathfrak{F} = (W, R)$, if there is some $w \in W$ and some variable assignment $\mathfrak{a} : \mathsf{Var} \to D$ such that $\mathfrak{M}, w \models^{\mathfrak{a}} \varphi$. We say that φ is *valid* in $\mathfrak{F}$ if its negation $\neg\varphi$ cannot be satisfied in any model based on $\mathfrak{F}$; in which case we write that $\mathfrak{F} \models \varphi$.

Given a non-empty class $\mathcal{C}$ of Kripke frames, we define the first-order modal logic of $\mathcal{C}$ to be the set

$$\mathsf{Q}^{\#}\mathsf{Log}(\mathcal{C}) = \{\varphi \in \mathcal{Q}^{\#}\mathcal{ML} \ : \ \mathfrak{F} \models \varphi \text{ for all } \mathfrak{F} \in \mathcal{C}\}$$

of all formulas that are valid in all first-order Kripke models based on frames belonging to $\mathcal{C}$. For a propositional modal logic L, we define $\mathbf{Q}^{\#}L = \mathsf{Q}^{\#}\mathsf{Log}(\mathsf{Fr}\ L)$, where $\mathsf{Fr}\ L$ denotes the class of all frames for L; *i.e.* those frames validating every formula of L.

We will be interested in the *decision (validity) problem* for $\mathbf{Q}^{\#}L$ which asks whether a given formula belongs to $\mathbf{Q}^{\#}L$, and is complementary to the *satisfiability problem* which asks whether a given formula is *satisfiable with respect to* $\mathbf{Q}^{\#}L$; *i.e.* satisfiable in a model based on some frame $\mathfrak{F} \in \mathsf{Fr}\ L$.

Definition 2.1 A first-order modal logic $\mathbf{Q}^{\#}L$ is said to have the *poly-size (resp. exponential) finite model property (fmp)* if every φ that is satisfiable with respect to $\mathbf{Q}^{\#}L$ can be satisfied in a finite model $\mathfrak{M} = (\mathfrak{F}, D, I)$ whose size is at most polynomial (resp. exponential) in the size of φ, with $\mathfrak{F} \in \mathsf{Fr}\ L$.

In what follows, we will be interested in the quantified versions of the propositional modal logics **K**, **KT**, **KB**, **S5**, and **Alt**, whose formulas are validated by the class of all frames, all reflexive frames, all symmetric frames, all equivalence relations, and all partial functions, respectively.

3 Main results

The main result of this section will be to show that the one-variable fragment of $\mathbf{Q}^{\#}\mathbf{K}$, characterised by the class of all frames, enjoys the exponential finite model property. Following this, the analogous results for the one-variable fragments of each of the logics $\mathbf{Q}^{\#}\mathbf{KT}$, $\mathbf{Q}^{\#}\mathbf{KB}$, $\mathbf{Q}^{\#}\mathbf{S5}$, and $\mathbf{Q}^{\#}\mathbf{Alt}$ can be obtained by reducing them to $\mathbf{Q}^{\#}\mathbf{K}$ (see Appendix A).

Theorem 3.1 *The fragment* $\mathbf{Q}^{\#}\mathbf{K} \cap \mathcal{Q}^{\#}\mathcal{ML}^1$ *has the exponential fmp.*

To prove this, we employ a version of the method of *quasimodels* [20,2]. Our quasimodels closely resemble full Kripke models, however, each first-order structure is replaced with a *quasistate*, each of which may be finitely represented. The basic structure of our quasimodels can still be infinite, and may require additional non-trivial 'pruning' techniques to ensure that large quasimodels can be reduced to smaller finite quasimodels without affecting satisfiability. Therein lies the crux of the problem we must solve.

First, let us fix some arbitrary first-order modal formula $\varphi \in \mathcal{Q}^{\#}\mathcal{ML}$, and throughout what follows let $n = |\mathrm{sub}(\varphi)|$ denote the number of subformulas of φ, $m = \mathsf{md}(\varphi)$ denote the modal depth of φ, and $C = \mathsf{count}(\varphi)$ denote the value of the largest quantifier subscript occurring in φ. In particular we note that $n, m < ||\varphi||$, while $C < 2^{||\varphi||}$, owing to the binary encoding of subscripts.

We define a *type* for φ to be any subset $t \subseteq \mathrm{sub}(\varphi)$ that is *Boolean-saturated* in the sense that:

(tp1) $\neg\psi \in t$ if and only if $\psi \notin t$, for all $\neg\psi \in \mathrm{sub}(\varphi)$, and

(tp2) $\psi_1 \wedge \psi_2 \in t$ if and only if $\psi_1 \in t$ and $\psi_2 \in t$, for all $\psi_1 \wedge \psi_2 \in \mathrm{sub}(\varphi)$.

Definition 3.2 We define a *quasistate* for φ to be a pair (T, μ) such that:

(qs1) T is a non-empty set of *types* for φ,

(qs2) $\mu : T \to \{1, \ldots, C, C+1\}$ is a *'multiplicity'* function,

(qs3) *($\exists_{\leq c}$-saturation)* For all $t \in T$ and $(\exists_{\leq c} x\ \xi) \in \mathrm{sub}(\varphi)$,

$$(\exists_{\leq c} x\ \xi) \in t \qquad \iff \qquad \sum_{t' \in T(\xi)} \mu(t') \ \leq \ c,$$

where $T(\xi) = \{t \in T \ : \ \xi \in t\}$ denotes the set of types belonging to T that contain ξ.

Note that the size of each quasistate cannot exceed the number of distinct types for φ, which is to say that $|T| \leq 2^n$. The multiplicity function indicates how many 'duplicates' of each type are required in order to transform the quasistate into an appropriate first-order structure. Note that φ is indifferent to any duplicates in excess of the value of its largest quantifier subscript.

A *basic structure* for φ is a triple $(W, \prec, \boldsymbol{q})$, where $(W, \prec)$ is an intransitive, irreflexive tree of depth $\leq m$, and $\boldsymbol{q}$ is a function associating each $w \in W$ with a quasistate $\boldsymbol{q}(w) = (T_w, \mu_w)$. An *(indexed) run through* $(W, \prec, \boldsymbol{q})$ is a pair $r = (f_r, i_r)$, where f_r is a function associating each $w \in W$ with a type $f_r(w) \in T_w$, and i_r is an index used to distinguish otherwise identical runs. For convenience, we do not distinguish between the run and the function described by its first argument, writing $r(w)$ in place of $f_r(w)$, for $w \in W$.

Definition 3.3 A *quasimodel for* φ is a tuple $\mathfrak{Q} = (W, \prec, \boldsymbol{q}, \mathfrak{R})$ such that:

(qm1) $(W, \prec, \boldsymbol{q})$ is a basic structure for φ, and $\mathfrak{R}$ is an set of indexed runs through $(W, \prec, \boldsymbol{q})$,

(qm2) There is some $w_0 \in W$ and $t_0 \in T_{w_0}$ such that $\varphi \in t_0$,

(qm3) *(coherence)* For all $r \in \mathfrak{R}$, $w \in W$ and $\Diamond\xi \in \mathrm{sub}(\varphi)$,

$$\exists v \in W;\ w \prec v \text{ and } \xi \in r(v) \quad \Longrightarrow \quad \Diamond\xi \in r(w),$$

(qm4) *(saturation)* For all $r \in \mathfrak{R}$, $w \in W$ and $\Diamond\xi \in \mathrm{sub}(\varphi)$,

$$\Diamond\xi \in r(w) \quad \Longrightarrow \quad \exists v \in W;\ w \prec v \text{ and } \xi \in r(v),$$

(qm5) For all $w \in W$ and $t \in T_w$,

$$\mu_w(t) = \min\big(\,|\{r \in \mathfrak{R} : r(w) = t\}|\,,\ C+1\big).$$

The *size* of $\mathfrak{Q}$ is taken to be $|W| \cdot |\mathfrak{R}|$.

The following lemma establishes that our quasimodels precisely capture the notion of satisfiability with respect to $\mathbf{Q}^{\#}\mathbf{K}$, and that every quasimodel for φ can be transformed into model for φ of proportional size.

Lemma 3.4 *Let $\varphi \in \mathcal{Q}^{\#}\mathcal{ML}^1$ be an arbitrary formula in one-variable. Then φ is satisfiable with respect to $\mathbf{Q}^{\#}\mathbf{K}$ iff there is a quasimodel for φ.*

Proof. Suppose that φ is satisfiable with respect to $\mathbf{Q}^{\#}\mathbf{K}$. Then $\mathfrak{M}, w_0 \models^{\mathfrak{a}} \varphi$ for some first-order Kripke model $\mathfrak{M} = (\mathfrak{F}, D, I)$, where $\mathfrak{F} = (W, \prec) \in \mathsf{Fr}\ \mathbf{K}$ is a frame for $\mathbf{K}$, with $w_0 \in W$. By a standard unravelling argument [1], we may assume without any loss of generality that $\mathfrak{F}$ is an intransitive, irreflexive tree of depth at most $m = \mathsf{md}(\varphi)$.

With each $w \in W$ and $a \in D$, we associate the type

$$\mathrm{tp}_w^{\mathfrak{M}}[a] = \{\xi \in \mathrm{sub}(\varphi) \ : \ \mathfrak{M}, w \models^{\mathfrak{a}(x/a)} \xi\},$$

and define a basic structure $(W, \prec, \boldsymbol{q})$, by taking $\boldsymbol{q}(w) = (T_w, \mu_w)$, for all $w \in W$, where

$$T_w = \{\mathrm{tp}_w^{\mathfrak{M}}[a] \ : \ a \in D\} \quad \text{and} \quad \mu_w(t) = \min\left(\left|\{a \in D : \mathrm{tp}_w^{\mathfrak{M}}[a] = t\}\right|, C+1\right)$$

for all $t \in T_w$. It is straightforward to check that $\boldsymbol{q}(w)$ is a quasistate, for each $w \in W$. Indeed, suppose that $\mathrm{tp}_w^{\mathfrak{M}}[a] \in T_w$ and that $(\exists_{\le c} x\ \xi) \in \mathrm{sub}(\varphi)$, for some $c \le C$, then we have that:

$$\begin{aligned} (\exists_{\le c} x\ \xi) \in \mathrm{tp}_w^{\mathfrak{M}}[a] \quad &\Longleftrightarrow \quad \mathfrak{M}, w \models^{\mathfrak{a}(x/a)} (\exists_{\le c} x\ \xi) \quad \text{by definition,} \\ &\Longleftrightarrow \quad \left|\{b \in D \ : \ \mathfrak{M}, w \models^{\mathfrak{a}(x/b)} \xi\}\right| \ \le \ c, \\ &\Longleftrightarrow \quad \sum_{t' \in T_w(\xi)} \left|\{b \in D \ : \ \mathrm{tp}_w^{\mathfrak{M}}[b] = t'\}\right| \ \le \ c, \\ &\Longleftrightarrow \quad \sum_{t' \in T_w(\xi)} \mu_w(t') \ \le \ c. \end{aligned}$$

The final equivalence follows from the fact that each summand strictly less than $(C+1)$, since $c \le C$. Hence, it follows from the definition that $\mu_w(t) = \left|\{b \in D \ : \ \mathrm{tp}_w^{\mathfrak{M}}[b] = t'\}\right|$, for all $t' \in T_w(\xi)$.
Furthermore, for each $a \in D$ we define an indexed run $r_a = (f_a, a)$, where $f_a : W \to 2^{\mathrm{sub}(\varphi)}$ is the function defined such that

$$f_a(w) = \mathrm{tp}_w^{\mathfrak{M}}[a],$$

for all $w \in W$. We then take $\mathfrak{R} = \{r_a \ : \ a \in D\}$ to be the set of all such indexed runs through $(W, \prec, \boldsymbol{q})$. Note that there may be many runs in $\mathfrak{R}$ that differ only in their index. It is straightforward to check that $(W, \prec, \boldsymbol{q}, \mathfrak{R})$ is a quasimodel for φ.

Conversely, suppose that $\mathfrak{Q} = (W, \prec, \boldsymbol{q}, \mathfrak{R})$ is a quasimodel for φ. We define a first-order Kripke model $\mathfrak{M} = (\mathfrak{F}, D, I)$, by taking

$$\mathfrak{F} = (W, \prec) \in \mathsf{Fr}\ \mathbf{K}, \qquad D = \mathfrak{R}, \qquad \text{and} \qquad P_i^{I(w)} = \{r \in \mathfrak{R} : P_i(x) \in r(w)\},$$

for all predicate symbols $P_i \in \mathsf{Pred}$ and $w \in W$. It remains to check that $\mathfrak{M}$ is a model for φ.

Claim 3.5 *We claim that, for all $w \in W$, $r \in \mathfrak{R}$, and $\psi \in \mathrm{sub}(\varphi)$,*

$$\mathfrak{M}, w \models^{\mathfrak{a}(x/r)} \psi \qquad \Longleftrightarrow \qquad \psi \in r(w).$$

This can be established by induction on the construction of ψ, the details for which can be found in Appendix B.

By **(qm2)**, there is some $w_0 \in W$ and $t_0 \in T_{w_0}$ such that $\varphi \in t_0$, while by **(qm5)** we have that there is some $r_0 \in \mathfrak{R}$ such that $r_0(w_0) = t_0$. Hence, it follows from (I.H.) that $\mathfrak{M}, w_0 \models^{\mathfrak{a}(x/r_0)} \varphi$, which is to say that φ is satisfiable with respect to $\mathbf{Q}^{\#}\mathbf{K}$, as required. □

Hence, to show that the one-variable fragment $\mathbf{Q}^{\#}\mathbf{K} \cap \mathcal{Q}^{\#}\mathcal{ML}^1$ has the exponential fmp, it is enough to show that every quasimodel for φ can be transformed into a finite quasimodel that is at most exponential in the size of φ.

Lemma 3.6 *If φ has a quasimodel, then φ has a quasimodel that is at most exponential in the size of φ.*

Proof. Suppose that $\mathfrak{Q} = (W, \prec, \boldsymbol{q}, \mathfrak{R})$ is a quasimodel for φ. The proof follows two stages: the first involves pruning both the basic structure and the set of runs so that they are both at most exponential in the size of φ. During this stage we inadvertently destroy some of the defining properties of our quasimodel; in particular the saturation condition **(qm4)**. In the second stage we remedy this deficiency by adding multiple 'copies' of each quasistate and performing 'surgery' on a finite set of runs to repair saturation.

Step 1) Firstly, it follows from **(qm2)** that there is some $w_0 \in W$ and $t_0 \in T_{w_0}$ such that $\varphi \in t_0$. By **(qm5)**, for each $w \in W$ and each $t \in T_w$ we may fix some run $s_{(w,t)} \in \mathfrak{R}$ such that $s_{(w,t)}(w) = t$. Take $\mathfrak{S}(w) = \{s_{(w,t)} : t \in T_w\}$ be to the set comprising all such runs, for each $w \in W$. In particular, we note that $|\mathfrak{S}(w)| = |T_w| \leq 2^n$. Furthermore, by **(qm4)**, for each $\Diamond\alpha \in t$ we may fix some $v = v_{(w,t,\alpha)} \in W$ such that $w \prec v$ and $\alpha \in s_{(w,t)}(v)$.

We define inductively a sequence of subsets $W_i \subseteq W$, for $i = 0, \dots, m$, by taking $W_0 = \{w_0\}$, and

$$W_{k+1} = \big\{v_{(w,t,\alpha)} \in W \ : \ w \in W_k,\ t \in T_w,\ \text{and } \Diamond\alpha \in t\big\},$$

for $k < m$. We then define a new basic structure $(W', \prec', \boldsymbol{q}')$, by taking

$$W' = \bigcup_{k=0}^{m} W_k \qquad u \prec' v \iff u \prec v, \qquad \text{and} \qquad \boldsymbol{q}'(u) = \boldsymbol{q}(u),$$

for all $u, v \in W'$.

Let $\mathfrak{S} = \bigcup\{\mathfrak{S}(w) : w \in W'\}$, and note that $\mathfrak{S}$ is finite since it is a finite union of finite sets of runs. However, $\mathfrak{S}$ need not be plentiful enough to accommodate **(qm5)**. Hence we must extend $\mathfrak{S}$ to a 'small' subset $\mathfrak{R}'$ of $\mathfrak{R}$ by choosing sufficiently many runs so as to satisfy **(qm5)**.

More precisely, for each $w \in W'$, $t \in T_w$ and $i < \mu_w(t)$ we can fix some $r_{(w,t,i)} \in \mathfrak{R}$ such that $r_{(w,t,i)}(w) = t$, and $r_{(w,t,i)} \neq r_{(w,t,j)}$ for $i \neq j$, which we are able to do since $\mathfrak{Q}$ satisfies **(qm5)**. Furthermore, we may assume without any loss of generality that $r_{(w,t,0)} = s_{(w,t)} \in \mathfrak{S}(w)$, as defined above. We take

$$\mathfrak{R}' = \left\{ r_{(w,t,i)} \in \mathfrak{R} \ : \ w \in W',\ t \in T_w \text{ and } i < \mu_w(t) \right\}$$

to be the set of all such runs, and define $\mathfrak{Q}' = (W', \prec', \boldsymbol{q}', \mathfrak{R}')$. We note that

$$|W'| \leq |W_0| + \cdots + |W_m| \leq (m+1) \cdot |W_m| \leq (m+1) \cdot (n \cdot 2^n)^m \qquad (1)$$

$$|\mathfrak{R}'| \leq |W'| \cdot \max_{w \in W} |T_w| \cdot (C+1) \leq (m+1) \cdot (n \cdot 2^n)^m \cdot 2^n \cdot (C+1) \qquad (2)$$

Furthermore, by construction, $\mathfrak{Q}'$ satisfies each of the conditions **(qm1)**, **(qm2)**, **(qm3)**, and **(qm5)**, as can be easily verified. However $\mathfrak{Q}'$ fails to satisfy the saturation condition **(qm4)**. To remedy this, we diverge from the techniques of [20,2] by extending our basic structure with not one but *multiple* 'copies' of each quasistate; each associated with a given transposition of runs.

Step 2) Let $\mathsf{Sym}(\mathfrak{R}')$ denote the set of all permutations $\sigma : \mathfrak{R}' \to \mathfrak{R}'$ on the set of runs $\mathfrak{R}'$, with $\mathsf{id} \in \mathsf{Sym}(\mathfrak{R}')$ denoting the identity function. For each $w \in W'$ and each $r \in \mathfrak{R}'$, let $\tau_{(w,r)} \in \mathsf{Sym}(\mathfrak{R}')$ denote the permutation that transposes r and $s_{(w,t)} \in \mathfrak{S}(w)$, where $t = r(w)$, and let $\mathsf{Trans}(w) = \{\tau_{(w,r)} : r \in \mathfrak{R}'\}$ denote the set of all such transpositions. In particular, we have that $|\mathsf{Trans}(w)| \leq |\mathfrak{R}'|$ is at most exponential in the size of φ.

In what follows, we construct a new basic structure based on some 'small' subset of $W' \times \mathsf{Sym}(\mathfrak{R}')$. Naturally, we cannot construct a basic structure out of the set of all pairs from $W' \times \mathsf{Sym}(\mathfrak{R}')$ if we are to insist on an exponential upper bound on the size of the quasimodel. Instead, for each $(w, \sigma) \in W' \times \mathsf{Sym}(\mathfrak{R}')$, we may define a small set of *successors* $S(w, \sigma) \subseteq W' \times \mathsf{Sym}(\mathfrak{R}')$, by taking

$$S(w, \sigma) = \{(v, \sigma') \ : \ w \prec v \text{ and } \sigma' = (\tau \circ \sigma) \text{ for some } \tau \in \mathsf{Trans}(w)\}.$$

In particular, we note that $|S(w, \sigma)| \leq |W'| \cdot |\mathsf{Trans}(w)| \leq |W'| \cdot |\mathfrak{R}'|$ is at most exponential in the size of φ. We construct a new sequence of sets $W'_i \subseteq W' \times \mathsf{Sym}(\mathfrak{R}')$, for $i = 0, \ldots, m$, by taking

$$W'_0 = \{(w_0, \mathsf{id})\} \qquad \text{and} \qquad W'_{k+1} = \bigcup \left\{ S(w, \sigma) \ : \ (w, \sigma) \in W'_k \right\}.$$

for $k < m$. The new basic structure is defined to be the triple $(W'', \prec'', \boldsymbol{q}'')$, where

$$W'' = \bigcup_{k=0}^{m} W'_k, \qquad (u, \sigma) \prec'' (v, \rho) \iff (v, \rho) \in S(u, \sigma)$$

and $\boldsymbol{q}''(u, \sigma) = \boldsymbol{q}'(u)$, for all $(u, \sigma), (v, \rho) \in W''$.

Finally, for each run $r \in \mathfrak{R}'$ we define a new run $\widehat{r}$ through $(W'', \prec'', \boldsymbol{q}'')$, by taking

$$\widehat{r}(w, \sigma) = \sigma(r)(w)$$

for all $(w, \sigma) \in W''$. That is to say that the new run $\widehat{r}$ behaves at $(w, \sigma) \in W''$ as $\sigma(r) \in \mathfrak{R}'$ does at $w \in W'$. Take $\mathfrak{R}'' = \{\widehat{r} : r \in \mathfrak{R}'\}$ to be the set of all such runs, and define $\mathfrak{Q}'' = (W'', \prec'', \boldsymbol{q}'', \mathfrak{R}'')$, where

$$|W''| \leq (m+1) \cdot (|W'| \cdot |\mathfrak{R}'|)^m \qquad \text{and} \qquad |\mathfrak{R}''| = |\mathfrak{R}'| \tag{3}$$

are both at most exponential in the size of φ. All that remains is to show that $\mathfrak{Q}''$ is a quasimodel for φ.

– It follows from the construction that $\varphi \in t_0$ for some $t_0 \in T_{w_0} = T_{(w_0, \mathsf{id})}$, where $(w_0, \mathsf{id}) \in W'$, as required for **(qm2)**.

– For **(qm3)**, suppose that $\widehat{r} \in \mathfrak{R}''$, $(w, \sigma), (v, \rho) \in W''$ and $\Diamond\alpha \in \text{sub}(\varphi)$ are such that $(w, \sigma) \prec'' (v, \rho)$ and $\alpha \in \widehat{r}(v, \rho)$.

  By definition we have that $(v, \rho) \in S(w, \sigma)$, which is to say that $w \prec v$ and $\rho = \tau \circ \sigma$ for some transposition $\tau \in \mathsf{Trans}(w)$. Hence we have that

$$\alpha \in \widehat{r}(v, \rho) \ = \ \rho(r)(v) \ = \ (\tau \circ \sigma)(r)(v) \ = \ \tau(\sigma(r))(v),$$

  where $\tau(\sigma(r)) \in \mathfrak{R}'$. Since $\mathfrak{Q}'$ is coherent and $w \prec' v$, we have that $\Diamond\alpha \in \tau(\sigma(r))(w)$. However, we have that $\tau \in \mathsf{Trans}(w)$ and hence by definition $\tau(\sigma(r))(w) = \sigma(r)(w)$, since τ transposes only runs that coincide at w. In particular, we have that $\Diamond\alpha \in \sigma(r)(w)$, which is to say that $\Diamond\alpha \in \widehat{r}(w, \sigma)$, as required.

– For **(qm4)**, suppose that $\widehat{r} \in \mathfrak{R}'$, $(w, \sigma) \in W''$ and $\Diamond\alpha \in \text{sub}(\varphi)$ are such that $\Diamond\alpha \in \widehat{r}(w, \sigma)$. This is to say that $\Diamond\alpha \in \sigma(r)(w)$. Let $t = \sigma(r)(w)$ and let $s_{(w,t)} \in \mathfrak{S}(w)$ be such that $s_{(w,t)}(w) = t$. By construction there is some $v = v_{(w,t,\alpha)} \in W'$ such that $w \prec' v$ and $\alpha \in s_{(w,t)}(v)$.

  Let $\tau = \tau_{(w,\sigma(r))} \in \mathsf{Trans}(w)$ be the transposition that swaps $\sigma(r) \in \mathfrak{R}'$ and $s_{(w,t)} \in \mathfrak{S}(w)$. It follows from the construction that there is some $(v, \tau \circ \sigma) \in S(w, \sigma) \subseteq W''$ such that

$$\alpha \ \in \ s_{(w,t)}(v) \ = \ \tau(\sigma(r))(v) \ = \ (\tau \circ \sigma)(r)(v) \ = \ \widehat{r}(v, \tau \circ \sigma),$$

  and $(w, \sigma) \prec'' (v, \tau \circ \sigma)$, as required.

– For **(qm5)**, suppose that $(w, \sigma) \in W''$ and $t \in T_{(w,\sigma)} = T_w$ and consider the following sets:

$$X = \{\widehat{r} \in \mathfrak{R}'' : \widehat{r}(w, \sigma) = t\} \qquad \text{and} \qquad Y = \{r \in \mathfrak{R}' : r(w) = t\}.$$

  We define a bijection $f : X \to Y$ by taking $f(\widehat{r}) = \sigma(r)$, for all $\widehat{r} \in X$, since by definition $\widehat{r}(w, \sigma) = \sigma(r)(w) = f(\widehat{r})(w)$. Hence $\widehat{r} \in X$ if and only if $f(\widehat{r}) \in Y$, and thus $|X| = |Y|$. That is to say that the number of runs passing through each type remains unaffected by Step 2 of our construction. It then follows from the definitions that

$$\mu_{(w,\sigma)}(t) \;=\; \mu_w(t) \;=\; \min(|Y|, C+1) \;=\; \min(|X|, C+1)$$

as required.

Hence we have established that $\mathfrak{Q}''$ is a quasimodel for φ, whose size is at most exponential in the size of φ, as can be deduced from (1)–(3), as required. □

Theorem 3.1 now follows from Lemmas 3.4–3.6, and hence the one-variable fragment $\mathbf{Q^{\#}K} \cap \mathcal{Q}^{\#}\mathcal{ML}^1$ has the exponential fmp. This provides us with an optimal upper bound on the complexity of the decision problem for this fragment; the lower bound being provided by the coNExpTime-hardness of the the regular (counting-free) one-variable fragment $\mathbf{Q^{\#}K} \cap \mathcal{Q}^{\#}\mathcal{ML}_0^1$ [10].

Corollary 3.7 *The one-variable fragment of* $\mathbf{Q^{\#}K}$ *is* coNExpTime*-complete.*

We note that the decision problems for each of the logics $\mathbf{Q^{\#}KT}$, $\mathbf{Q^{\#}KB}$, $\mathbf{Q^{\#}S5}$ and $\mathbf{Q^{\#}Alt}$ can be polynomially reduced to that of $\mathbf{Q^{\#}K}$, such that the exponential fmp is preserved. Hence, we are able to deduce that the one-variable fragments of each of these logics also enjoys the exponential finite model property.

Note that the existence of a polynomial reduction between two *propositional* modal logics does not guarantee an analogous reduction between their first-order counterparts, without a complementary model transformation. (see, for example, [2, Remark 6.19]). However, in the aforementioned cases, such 'model-level' reductions are not hard to construct; examples of which are sketched in Appendix A.

Corollary 3.8 *Each of the fragments* $L \cap \mathcal{Q}^{\#}\mathcal{ML}^1$ *has the exponential fmp, for* $L \in \{\mathbf{Q^{\#}KT},\ \mathbf{Q^{\#}KB},\ \mathbf{Q^{\#}S5},\ \mathbf{Q^{\#}Alt}\}$.

It is clear that we can do no better than this bound, since even the classical one-variable fragment with counting quantifiers contains 'small' formulas such as $\varphi_k = \exists_{>2^k} x\ \top$, whose size is linear in k but can only be satisfied in models containing at least 2^k elements, where $\top := P_0(x) \vee \neg P_0(x)$. Indeed, even without counting quantifiers, each of the logics $L \cap \mathcal{Q}^{\#}\mathcal{ML}_0^1$, for $L \in \{\mathbf{Q^{\#}K}, \mathbf{Q^{\#}KT}, \mathbf{Q^{\#}KB}, \mathbf{Q^{\#}S5}\}$ admit formulas that cannot be satisfied in any 'small' model.

On the other hand, it is known that $\mathbf{Q^{\#}Alt} \cap \mathcal{Q}^{\#}\mathcal{ML}_0^1$ enjoys the poly-size model property, and is therefore coNP-complete [10]. In fact, it can be shown that the decision problem for $\mathbf{Q^{\#}Alt} \cap \mathcal{Q}^{\#}\mathcal{ML}^1$ is also coNP-complete, by a reduction to the one-variable fragment of classical first-order logic with counting quantifiers, despite lacking the poly-size model property. The following result follows the approach taken in [2, Proposition 5.35].

Theorem 3.9 *The one-variable fragment of* $\mathbf{Q^{\#}Alt}$ *is* coNP*-complete.*

Proof. Let $\varphi \in \mathcal{Q}^{\#}\mathcal{ML}^{1}$ be an arbitrary formula in one variable with quantifier. For each $\psi \in \mathrm{sub}(\varphi)$, let $Q^{0}_{\psi}, \ldots, Q^{m+1}_{\psi} \in \mathsf{Pred}$ be fresh monadic predicate symbols not occurring in φ, where $m = \mathsf{md}(\varphi)$. For each $i \leq m$, let ζ_i denote the conjunction of the following formulas:

$$\bigwedge_{(\psi_1 \wedge \psi_2) \in \mathrm{sub}(\varphi)} Q^{i}_{\psi_1 \wedge \psi_2}(x) \leftrightarrow \big(Q^{i}_{\psi_1}(x) \wedge Q^{i}_{\psi_2}(x)\big),$$
$$\bigwedge_{\neg\psi \in \mathrm{sub}(\varphi)} Q^{i}_{\neg\psi}(x) \leftrightarrow \neg Q^{i}_{\psi}(x),$$
$$\bigwedge_{(\exists_{=c} x\ \psi) \in \mathrm{sub}(\varphi)} Q^{i}_{(\exists_{=c} x\ \psi)}(x) \leftrightarrow \exists_{=c} x\ Q^{i}_{\psi}(x),$$
$$\bigwedge_{\Diamond\psi \in \mathrm{sub}(\varphi)} Q^{i}_{\Diamond\psi}(x) \leftrightarrow Q^{i+1}_{\psi}(x)$$

and let $\mathsf{t}_\ell(\varphi) = \bigwedge_{i \leq \ell} \forall x \zeta_i \to Q^{0}_{\varphi}$. In particular, we note that the size of each $\mathsf{t}_\ell(\varphi)$ is at most polynomial in the size of φ. We claim that $\varphi \in \mathbf{Q^{\#}Alt}$ if and only if $\mathsf{t}_\ell(\varphi)$ is valid with respect to (classical) first-order logic, with counting quantifiers, for all $\ell \leq m$.

($\Rightarrow$) Suppose that there is some $\ell \leq m$ such that $\mathsf{t}_\ell(\varphi)$ is not valid. Then $\mathfrak{A} \models^{\mathfrak{a}} \bigwedge_{i \leq \ell} \forall x \zeta_i$ and $\mathfrak{A} \not\models^{\mathfrak{a}} Q^{0}_{\varphi}$ for some classical first-order model $\mathfrak{A} = (D, J)$. We define a first-order Kripke model $\mathfrak{M} = (\mathfrak{F}, D, I)$, where $\mathfrak{F} = (W, R)$, by taking $W = \{w_i : i \leq \ell\}$, $R = \{(w_i, w_{i+1}) : i < \ell\}$, and $P^{I(w_i)} = (Q^{i}_{P(x)})^{J}$, for all predicate symbols $P \in \mathsf{Pred}$ occurring in φ, and all $w_i \in W$. We claim that, for all $w_i \in W$, $b \in D$ and $\psi \in \mathrm{sub}(\varphi)$,

$$\mathfrak{M}, w_i \models^{\mathfrak{a}(x/b)} \psi \quad \Longleftrightarrow \quad \mathfrak{A} \models^{\mathfrak{a}(x/b)} Q^{i}_{\psi}$$

as can be verified by induction on the construction of ψ. Hence it follows that $\mathfrak{M}, w_0 \not\models^{\mathfrak{a}} \varphi$, which is to say that $\varphi \notin \mathbf{Q^{\#}Alt}$, as required.

($\Leftarrow$) Suppose that $\varphi \notin \mathbf{Q^{\#}Alt}$. Then $\mathfrak{M}, w \models^{\mathfrak{a}} \varphi$ for some first-order Kripke model $\mathfrak{M} = (\mathfrak{F}, D, I)$, where $\mathfrak{F} = (W, R) \in \mathsf{Fr}\ \mathbf{Alt}$. Without loss of generality we may suppose that $W = \{w_i : i \leq \ell\}$ and $R = \{(w_i, w_{i+1}) : i < \ell\}$, for some $\ell \leq m$. We define a classical first-order model $\mathfrak{A} = (D, J)$ by taking

$$(Q^{i}_{\psi})^{J} = \{b \in D\ :\ \mathfrak{M}, w_i \models^{\mathfrak{a}(x/b)} \psi\}$$

for $\psi \in \mathrm{sub}(\varphi)$ and $i \leq \ell$. It then follows from this definition that $\mathfrak{A} \not\models^{\mathfrak{a}} \mathsf{t}_\ell(\varphi)$, which is to say that there is some $\mathsf{t}_\ell(\varphi)$ is not valid, for $\ell \leq m$.

Since the validity problem for the one-variable fragment of (classical) first-order logic, with counting quantifiers, is decidable in coNP [14], so too must be that of the one-variable fragment of $\mathbf{Q^{\#}Alt}$, as required. □

Finally, we note that Theorem 3.9 holds if we replace $\mathbf{Q^{\#}Alt}$ with its *serial* [1] extension $\mathbf{Q^{\#}AltD}$. We need only add reflexive loop to the last element of $\mathfrak{F}$.

[1] A frame (W, R) is said to be serial if for every $w \in W$ there is some $v \in W$ such that wRv.

4 Two-dimensional Modal Logics

First-order modal logics are intimately related to another extensively studied formalism; that of many-dimensional modal logics [17,15,2,9,11]. Given a countably infinite set of propositional variables $\mathsf{Prop} = \{p_0, p_1, \dots\}$, let $\mathcal{ML}_2$ denote the set of bimodal formulas defined by the following grammar:

$$\varphi \ ::= \ p_i \ \mid \ \neg\varphi \ \mid \ (\varphi_1 \wedge \varphi_2) \ \mid \ \Diamond_h \varphi \ \mid \ \Diamond_v \varphi$$

where $p_i \in \mathsf{Prop}$.

Formulas of $\mathcal{ML}_2$ are interpreted over Kripke models $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$, where $\mathfrak{F} = (W, R_h, R_v)$ is a *bimodal Kripke frame*, with $R_h, R_v \subseteq W \times W$, and $\mathfrak{V} : \mathsf{Prop} \to 2^W$ is a propositional valuation. Satisfiability is defined in the usual way with $\Diamond_j \varphi$ being interpreted by the relation R_j, for $j = h, v$.

Of particular interest are product models in which the two modal operators act orthogonally. We define the *product* of two unimodal frames $\mathfrak{F}_h = (W_h, R_h)$ and $\mathfrak{F}_v = (W_v, R_v)$ to be the bimodal frame $\mathfrak{F}_h \times \mathfrak{F}_v = (W_h \times W_v, \overline{R}_h, \overline{R}_v)$, where

$$(u, v)\overline{R}_h(u', v') \iff uR_h u' \text{ and } v = v',$$

$$(u, v)\overline{R}_v(u', v') \iff u = u' \text{ and } vR_v v',$$

for all $u, u' \in W_h$ and $v, v' \in W_v$. The *product* of two unimodal logics L_h and L_v is defined to be the bimodal logic

$$L_h \times L_v = \{\varphi \in \mathcal{ML}_2 \ : \ \mathfrak{F}_h \times \mathfrak{F}_v \models \varphi \text{ where } \mathfrak{F}_j \in \mathsf{Fr}\ L_j \text{ for } j = h, v\}$$

Although product logics are characterised by their product frames, in general there may be frames for product logics that are not product frames. Moreover, it is not always obvious whether an arbitrary bimodal frame is a frame for a given product logic, if the logic happens to be non-finitely axiomatisable. For this reason, it is often more convenient to work with the following, more restrictive, variant of the general ('abstract') fmp.

Definition 4.1 A product logic $L_h \times L_v$ is said to have the *poly-size (resp. exponential) product fmp* if every $\varphi \notin L_h \times L_v$ can be refuted in a model based on a *product frame* for $L_h \times L_v$ that is at most polynomial (resp. exponential) in the size of φ.

Clearly, any logic possessing the product fmp also enjoys the more general 'abstract' fmp.

It is well established that products of the form $L \times \mathbf{S5}$ can be interpreted as syntactic variants of the one-variable fragment of first-order modal logic $\mathbf{Q}^{\#}L \cap \mathcal{Q}^{\#}\mathcal{ML}_0^1$, in which only quantifiers with zero subscripts are permitted [3]. A further connection can be established between products of the form $L \times \mathbf{Diff}$ and the one-variable fragment $\mathbf{Q}^{\#}L \cap \mathcal{Q}^{\#}\mathcal{ML}_1^1$, where **Diff** denotes von Wright's logic of 'elsewhere' [19,16], characterised by all those unimodal formulas that are valid in all *difference frames* of the form $(W, \neq)$.

We define the translation $(\cdot)^\dagger : \mathcal{ML}_2 \to \mathcal{Q}^\#\mathcal{ML}_1^1$ by taking

$$p_i^\dagger = P_i(x), \qquad (\neg\psi)^\dagger = \neg\psi^\dagger, \qquad (\psi_1 \wedge \psi_2)^\dagger = \psi_1^\dagger \wedge \psi_2^\dagger,$$
$$(\Diamond_h\psi)^\dagger = \Diamond\psi^\dagger, \qquad (\Diamond_v\psi)^\dagger = \exists^{\neq}x\ \psi^\dagger,$$

where $P_i \in \mathsf{Pred}$ is a unique monadic predicate associated with each propositional variable $p_i \in \mathsf{Prop}$, and $\exists^{\neq}x\ \varphi := \big(\neg\varphi \wedge \exists_{>0}x\ \varphi\big) \vee \exists_{>1}x\ \varphi$. Furthermore, with each first-order Kripke model $\mathfrak{M} = (\mathfrak{F}, D, I)$, we associate a propositional product model $\mathfrak{M}^\star = (\mathfrak{F}_h \times \mathfrak{F}_v, \mathfrak{V})$, by taking

$$\mathfrak{F}_h = \mathfrak{F} \in \mathsf{Fr}\ L, \qquad \mathfrak{F}_v = (D, \neq) \in \mathsf{Fr}\ \mathbf{Diff}, \qquad \mathfrak{V}(p_i) = \big\{(w, v) : v \in P_i^{I(w)}\big\}$$

for all $p_i \in \mathsf{Prop}$. It is then straightforward to check that φ is satisfiable in $\mathfrak{M}^\star$ if and only if $\varphi^\dagger$ is satisfiable in $\mathfrak{M}$. Furthermore, since the model transformation $(\cdot)^\star$ maps bijectively onto the class of product frames characterising $L \times \mathbf{Diff}$, we have the following polynomial reduction from $L \times \mathbf{Diff}$ to $\mathbf{Q}^\# L \cap \mathcal{Q}^\#\mathcal{ML}_1^1$.

Proposition 4.2 *$\varphi \in L \times \mathbf{Diff}$ if and only if $\varphi^\dagger \in \mathbf{Q}^\# L \cap \mathcal{Q}^\#\mathcal{ML}_1^1$.*

That is to say that products of the form $L \times \mathbf{Diff}$ can be embedded within the one-variable fragment $\mathbf{Q}^\# L \cap \mathcal{Q}^\#\mathcal{ML}_1^1$, with quantifier subscripts not exceeding one. Moreover, it follows that $\mathbf{Q}^\# L \cap \mathcal{Q}^\#\mathcal{ML}_1^1$ has the poly-size (resp. exponential) fmp if and only if $L \times \mathbf{Diff}$ has the poly-size (resp. exponential) *product* fmp. Consequently, we have the following corollary of Theorem 3.1.

Corollary 4.3 *Let $L \in \{\mathbf{K}, \mathbf{KT}, \mathbf{KB}, \mathbf{S5}\}$. Then*

(i) $L \times \mathbf{Diff}$ has the exponential product fmp,

(ii) The decision problem for $L \times \mathbf{Diff}$ is coNExpTime*-complete.*

The lower bounds follow from [10, Theorem 3.2], in which every bimodal logic between $\mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5}$ is shown to be coNExpTime-hard.

Previously, only non-elementary upper bounds on the complexity of $\mathbf{K} \times \mathbf{Diff}$ were known. In particular, $\mathbf{K} \times \mathbf{Diff}$ can be embedded into the product $\mathbf{K} \times \mathbf{Lin}$, whose decidability can be established by a variant on the 'mosaic' approach [20,2]. More recently, the *abstract* fmp of $\mathbf{K} \times \mathbf{Diff}$ had been established via a method of 'canonical filtrations' [18]. However, this yields only a non-elementary bound on the size of the filtrated models. Furthermore, it is known that $\mathbf{K} \times \mathbf{Diff}$ is non-finitely axiomatisable [5], and there is currently no known procedure for deciding whether an arbitrary bimodal frame is a frame for $\mathbf{K} \times \mathbf{Diff}$. Consequently, we cannot construct a feasible decision procedure from the abstract fmp alone.

It should be noted that the upper bound on the decision problem for $\mathbf{S5} \times \mathbf{Diff}$ follows from the coNExpTime-completeness of the two-variable fragment $\mathcal{C}_1^2$ with quantifier subscripts not exceeding one [12]. For instance, take $(\cdot)^\ddagger : \mathcal{ML}_2 \to \mathcal{C}^2$ to be the translation given:

$$p_i^\ddagger = P_i(x,y), \qquad (\neg\psi)^\ddagger = \neg\psi^\ddagger, \qquad (\psi_1 \wedge \psi_2)^\ddagger = \psi_1^\ddagger \wedge \psi_2^\ddagger,$$
$$(\Diamond_h \psi)^\ddagger = \exists_{>0} x\ \psi^\ddagger, \qquad (\Diamond_v \psi)^\ddagger = \exists^{\neq} y\ \big(D(y) \wedge \psi^\ddagger\big),$$

where $P_i \in \mathsf{Pred}$ is a unique binary predicate symbol associated with each propositional variable $p_i \in \mathsf{Prop}$ and $D \in \mathsf{Pred}$ is an auxiliary monadic predicate symbol, providing a guard on the $\exists^{\neq} y$ quantifier, defined above. We have that $\varphi \in \mathbf{S5} \times \mathbf{Diff}$ if and only if $\varphi^\ddagger$ is valid with respect to classical first-order logic.

Note, however, that the full two-variable fragment $\mathcal{C}^2$, with counting quantifiers, does not enjoy the finite model property, even if we restrict the quantifier subscripts to ≤ 1; as evidenced by the following formulas of [4]:

$$\forall x \exists y\ P(x,y)\ \wedge\ \forall y \exists_{\leq 1} x\ P(x,y)\ \wedge\ \exists y \forall x\ \neg P(x,y).$$

Hence, we cannot infer the exponential fmp of $\mathbf{S5} \times \mathbf{Diff}$ by appealing to the classical two-variable fragment, and that $(\cdot)^\ddagger$ maps onto a proper fragment of $\mathcal{C}^2$ possessing the exponential fmp.

Finally, since $\mathbf{Alt} \times \mathbf{Diff}$ is reducible to the fragment $\mathbf{Q}^{\#}\mathbf{Alt} \cap \mathcal{Q}^{\#}\mathcal{ML}_1^1$, we note the following corollary of Theorem 3.9.

Corollary 4.4 *The decision problem for* $\mathbf{Alt} \times \mathbf{Diff}$ *is* coNP*-complete.*

5 Discussion

We conclude with a discussion of several related open problems:

- It remains open whether these results can be extended to the *monodic* fragment, appropriately extended to counting quantifiers. However, it should be noted that there is no immediate application of the techniques developed in [21], which are able to prove that the (counting-free) monodic fragment of $\mathbf{Q}^{\#}\mathbf{K}^*$ is decidable, since even the one-variable fragment $\mathbf{Q}^{\#}\mathbf{K}^* \cap \mathcal{Q}^{\#}\mathcal{ML}_1^1$ is known to be non-recursively enumerable [6]. Here, $\mathbf{K}^*$ denotes the bimodal logic of all frames whose second relation is the transitive closure of the first.
- It is known that the one-variable fragment $\mathbf{Q}^{\#}\mathbf{K4} \cap \mathcal{Q}^{\#}\mathcal{ML}_0^1$, with the sole quantifier $\exists_{\leq 0} x$, has the fmp and is decidable in coN2ExpTime [3,2], where $\mathbf{K4}$ denotes the logic characterised by the class of all transitive frames. However, it remains open whether the full one-variable fragment $\mathbf{Q}^{\#}\mathbf{K4} \cap \mathcal{Q}^{\#}\mathcal{ML}^1$ is decidable or has even the 'abstract' fmp.
- For many first-order modal logics characterised by *linear orders*, the addition of counting quantifiers causes a jump from decidability to undecidability [7]. In particular the fragment $\mathbf{Q}^{\#}\mathbf{K4.3} \cap \mathcal{Q}^{\#}\mathcal{ML}_1^1$ is known to be undecidable, where $\mathbf{K4.3}$ denotes the logic of all linear orders. Moreover, the same is true if we consider the sub-logic characterised by models with *decreasing* domains. However, it remains open whether the same result holds in the case of *expanding* domains.

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A Modal Reductions

In this appendix we outline the reductions between the first-order modal logic $\mathbf{Q^{\#}K}$ and the logics $\mathbf{Q^{\#}KT}$, $\mathbf{Q^{\#}KB}$, $\mathbf{Q^{\#}S5}$ and $\mathbf{Q^{\#}Alt}$, discussed above in Section 3. Each of the reductions respects the number of first-order variables, thereby completing the proof of Corollary 3.8.

Let $\varphi \in \mathcal{Q}^{\#}\mathcal{ML}^1$ be an arbitrary first-order modal formula, and for each $\alpha \in \mathrm{sub}(\varphi)$, let Q_α be a fresh propositional variable not occurring in φ. We define a translation $(\cdot)^\dagger : \mathrm{sub}(\varphi) \to \mathcal{Q}^{\#}\mathcal{ML}^1$ by taking

$$P_i(x)^\dagger = P_i(x), \qquad (\neg\psi)^\dagger = \neg\psi^\dagger, \qquad (\psi_1 \wedge \psi_2)^\dagger = \psi_1^\dagger \wedge \psi_2^\dagger,$$
$$(\Diamond\psi)^\dagger = Q_\psi(x), \qquad (\exists_{\leq c} x\ \psi)^\dagger = \exists_{\leq c} x\ \psi^\dagger.$$

That is to say that we replace each subformula of the form $\Diamond\psi$ with a monadic predicate $Q_\psi(x)$. We then define the following formulas:

$$\zeta_1 := \bigwedge_{\xi \in \mathrm{sub}(\varphi)} \Box^{(m)}\big(Q_\xi(x) \to \Box\xi^\dagger\big) \ \wedge\ \Box^{(m)}\big(\Diamond\xi^\dagger \to Q_\xi(x)\big) \tag{A.1}$$

$$\zeta_2 := \bigwedge_{\xi \in \mathrm{sub}(\varphi)} \Box^{(m)}\big(Q_\xi(x) \leftrightarrow (\xi^\dagger \vee \Diamond\xi^\dagger)\big) \tag{A.2}$$

$$\zeta_3 := \bigwedge_{\xi \in \mathrm{sub}(\varphi)} \big(Q_\xi(x) \to \Box Q_\xi(x)\big) \ \wedge\ \big(\Diamond Q_\xi(x) \to Q_\xi(x)\big) \ \wedge\ \big(Q_\xi(x) \leftrightarrow (\xi^\dagger \vee \Diamond\xi^\dagger)\big) \tag{A.3}$$

$$\zeta_4 := \bigwedge_{\xi \in \mathrm{sub}(\varphi)} \Box^{(m)}\big(\Diamond\xi^\dagger \to Q_\xi(x)\big) \ \wedge\ \Box^{(m)}\big(\xi^\dagger \to \Box Q_\xi(x)\big) \ \wedge\ \Box^{(m)}\big(\Diamond(Q_\xi(x) \wedge \neg\Diamond\xi^\dagger) \to \xi^\dagger\big) \ \wedge\ \big(Q_\xi(x) \to \Diamond\xi^\dagger\big) \tag{A.4}$$

where $\Box^{(0)}\varphi := \varphi$ and $\Box^{(k)}\varphi := \varphi \wedge \Box\Box^{(k-1)}\varphi$, for $k > 0$.

Note that each of the formulas $\zeta_1, \zeta_2, \zeta_3$ and ζ_4, are at most linear in the size of φ, since $||\xi^\dagger|| \leq ||\xi||$, for all $\xi \in \mathrm{sub}(\varphi)$.

Proposition A.1 *Let $\varphi \in \mathcal{Q}^{\#}\mathcal{ML}^1$ and let $\zeta_1, \zeta_2, \zeta_3$, and ζ_4 be as above, then:*

(i) $\varphi \in \mathbf{Q^{\#}Alt}$ *if and only if* $(\forall x\zeta_1 \to \varphi^\dagger) \in \mathbf{Q^{\#}K}$,

(ii) $\varphi \in \mathbf{Q^{\#}KT}$ *if and only if* $(\forall x\zeta_2 \to \varphi^\dagger) \in \mathbf{Q^{\#}K}$,

(iii) $\varphi \in \mathbf{Q^{\#}S5}$ *if and only if* $(\forall x\zeta_3 \to \varphi^\dagger) \in \mathbf{Q^{\#}K}$,

(iv) $\varphi \in \mathbf{Q^{\#}KB}$ *if and only if* $(\forall x\zeta_4 \to \varphi^\dagger) \in \mathbf{Q^{\#}K}$.

Proof.

(i) ($\Rightarrow$) Suppose that $(\forall x\zeta_1 \to \varphi^\dagger) \notin \mathbf{Q^{\#}K}$. Then $\mathfrak{M}, r \models^{\mathfrak{a}} \forall x\zeta_1$ and $\mathfrak{M}, r \not\models^{\mathfrak{a}} \varphi$ for some model $\mathfrak{M} = (\mathfrak{F}, D, I)$, where $\mathfrak{F} = (W, R) \in \mathsf{Fr}\ \mathbf{K}$. Without loss of generality we may suppose that $\mathfrak{F}$ is an irreflexive, intransitive tree, rooted at $r \in W$. Let $w_0 R w_1 R \ldots R w_\ell$ denote the longest R-chain in $\mathfrak{F}$ such that

$w_0 = r$ and $\ell \leq m$, and define a new model $\mathfrak{M}' = (\mathfrak{F}', D, I)$ where $\mathfrak{F}' = (W', R') \in \mathsf{Fr}\ \mathbf{Alt}$, by taking $W' = \{w_i : i \leq \ell\}$ and $R' = R \cap (W' \times W')$. We claim that, for all $w \in W'$, $b \in D$, and $\psi \in \mathrm{sub}(\varphi)$,

$$\mathfrak{M}', w \models^{\mathfrak{a}(x/b)} \psi \quad \iff \quad \mathfrak{M}, w \models^{\mathfrak{a}(x/b)} \psi^{\dagger},$$

whenever $\mathsf{md}(\psi) + d(r, w) \leq m$, as can be verified by induction on the construction of ψ, where $d(r, w)$ denotes the distance[2] between r and w.

Hence it follows that $\mathfrak{M}', r \not\models^{\mathfrak{a}} \varphi$, which is to say that $\varphi \notin \mathbf{Q}^{\#}\mathbf{Alt}$, as required.

($\Leftarrow$) Conversely, suppose that $\varphi \notin \mathbf{Q}^{\#}\mathbf{Alt}$. Then $\mathfrak{M}, r \not\models^{\mathfrak{a}} \varphi$ for some model $\mathfrak{M} = (\mathfrak{F}, D, I)$, where $\mathfrak{F} = (W, R) \in \mathsf{Fr}\ \mathbf{Alt} \subseteq \mathsf{Fr}\ \mathbf{K}$ describes a partial function. We define a new model $\mathfrak{M}' = (\mathfrak{F}, D, I')$ by taking $P_i^{I'(w)} = P_i^{I(w)}$ for all $P_i(x) \in \mathrm{sub}(\varphi)$ and $b \in Q_\xi^{I'(w)}$ iff $\mathfrak{M}, w \models^{\mathfrak{a}(x/b)} \Diamond\xi$. We claim that, for all $w \in W$, $b \in D$, and $\psi \in \mathrm{sub}(\varphi)$,

$$\mathfrak{M}', w \models^{\mathfrak{a}(x/b)} \psi^{\dagger} \quad \iff \quad \mathfrak{M}, w \models^{\mathfrak{a}(x/b)} \psi,$$

as can be verified by induction on the construction of ψ.

Furthermore, since $\mathfrak{F}$ is a frame for $\mathbf{Alt}$, we have that $\mathfrak{M}, r \models^{\mathfrak{a}} \forall x \zeta_1$, as can be easily verified. Hence it follows that $\mathfrak{M}', r \not\models^{\mathfrak{a}} (\forall x \zeta_1 \to \varphi)$, which is to say that $(\forall x \zeta_1 \to \varphi) \notin \mathbf{Q}^{\#}\mathbf{K}$, as required.

(ii) ($\Rightarrow$) Suppose that $(\forall x \zeta_2 \to \varphi^{\dagger}) \notin \mathbf{Q}^{\#}\mathbf{K}$. Then $\mathfrak{M}, r \models^{\mathfrak{a}} \forall x \zeta_2$ and $\mathfrak{M}, r \not\models^{\mathfrak{a}} \varphi$ for some model $\mathfrak{M} = (\mathfrak{F}, D, I)$, where $\mathfrak{F} = (W, R) \in \mathsf{Fr}\ \mathbf{K}$. Without loss of generality we may suppose that $\mathfrak{F}$ is an irreflexive, intransitive tree, rooted at $r \in W$. Let $\mathfrak{F}^+ = (W, R^+) \in \mathsf{Fr}\ \mathbf{KT}$ denote the reflexive closure of $\mathfrak{F}$, where $R^+ = R \cup \{(w, w) : w \in W\}$, and let $\mathfrak{M}' = (\mathfrak{F}^+, D, I)$ be a new model. We claim that, for all $w \in W'$, $b \in D$, and $\psi \in \mathrm{sub}(\varphi)$,

$$\mathfrak{M}', w \models^{\mathfrak{a}(x/b)} \psi \quad \iff \quad \mathfrak{M}, w \models^{\mathfrak{a}(x/b)} \psi^{\dagger},$$

whenever $\mathsf{md}(\psi) + d(r, w) \leq m$, as can be verified by induction on the construction of ψ, where $d(r, w)$ denotes the distance between r and w.

Hence it follows that $\mathfrak{M}', r \not\models^{\mathfrak{a}} \varphi$, which is to say that $\varphi \notin \mathbf{Q}^{\#}\mathbf{KT}$, as required.

($\Leftarrow$) The converse direction is similar to that of (i), using the fact that that $\mathsf{Fr}\ \mathbf{KT} \subseteq \mathsf{Fr}\ \mathbf{K}$, and verifying that the new model satisfies $\forall x \zeta_2$, which follows from the structure of the frames for $\mathbf{KT}$.

(iii) ($\Rightarrow$) Suppose that $(\forall x \zeta_3 \to \varphi^{\dagger}) \notin \mathbf{Q}^{\#}\mathbf{K}$. Then $\mathfrak{M}, w \models^{\mathfrak{a}} \forall x \zeta_3$ and $\mathfrak{M}, w \not\models^{\mathfrak{a}} \varphi$ for some model $\mathfrak{M} = (\mathfrak{F}, D, I)$, where $\mathfrak{F} = (W, R) \in \mathsf{Fr}\ \mathbf{K}$. Without loss of

[2] The (geodesic) distance $d(w, v)$ is the smallest integer $n < \omega$ such that $w_0 R w_1 R \ldots R w_n$ is an R-chain, with $w_0 = w$ and $w_n = v$.

generality we may suppose that $\mathfrak{F}$ is an irreflexive, intransitive tree of depth ≤ 1, since both $\mathsf{md}(\zeta_3), \mathsf{md}(\varphi^\dagger) \leq 1$, rooted at $r \in W$. Let $\mathfrak{F}^\circ = (W, W \times W)$ denote the universal closure of $\mathfrak{F}$, and let $\mathfrak{M}' = (\mathfrak{F}^\circ, D, I)$ be a new model. We claim that, for all $w \in W$, $b \in D$, and $\psi \in \mathrm{sub}(\varphi)$,

$$\mathfrak{M}', w \models^{\mathfrak{a}(x/b)} \psi \quad \iff \quad \mathfrak{M}, w \models^{\mathfrak{a}(x/b)} \psi^\dagger,$$

as can be verified by induction on the construction of ψ.

Hence it follows that $\mathfrak{M}', r \not\models^{\mathfrak{a}} \varphi$, which is to say that $\varphi \notin \mathbf{Q^{\#}S5}$, as required.

($\Leftarrow$) Again, the converse direction is similar to that of (i), using the fact that that $\mathsf{Fr}\ \mathbf{S5} \subseteq \mathsf{Fr}\ \mathbf{K}$, and verifying that the new model satisfies $\forall x \zeta_3$, which follows from the structure of the frames for **S5**.

(iv) ($\Rightarrow$) Suppose that $(\forall x \zeta_4 \to \varphi^\dagger) \notin \mathbf{Q^{\#}K}$. Then $\mathfrak{M}, w \models^{\mathfrak{a}} \forall x \zeta_4$ and $\mathfrak{M}, w \not\models^{\mathfrak{a}} \varphi$ for some model $\mathfrak{M} = (\mathfrak{F}, D, I)$, where $\mathfrak{F} = (W, R) \in \mathsf{Fr}\ \mathbf{K}$. Without loss of generality we may suppose that $\mathfrak{F}$ is an irreflexive, intransitive tree of depth, rooted at $r \in W$. Let $\mathfrak{F}^\smile = (W, R \cup R^\smile) \in \mathsf{Fr}\ \mathbf{KB}$ denote the symmetric closure of $\mathfrak{F}$, where $R^\smile = \{(v, w) : (w, v) \in R\}$, and let $\mathfrak{M}' = (\mathfrak{F}^\smile, D, I)$ be a new model. We claim that, for $w \in W$, $b \in D$, and $\psi \in \mathrm{sub}(\varphi)$,

$$\mathfrak{M}', w \models^{\mathfrak{a}(x/b)} \psi \quad \iff \quad \mathfrak{M}, w \models^{\mathfrak{a}(x/b)} \psi^\dagger,$$

whenever $\mathsf{md}(\psi) + d(r, w) \leq m$, as can be verified by induction on the construction of ψ, where $d(r, w)$ denotes the distance between r and w.

Hence it follows that $\mathfrak{M}', r \not\models^{\mathfrak{a}} \varphi$, which is to say that $\varphi \notin \mathbf{Q^{\#}KB}$, as required.

($\Leftarrow$) The converse direction is similar to that of (i), using the fact that that $\mathsf{Fr}\ \mathbf{KB} \subseteq \mathsf{Fr}\ \mathbf{K}$, and verifying that the new model satisfies $\forall x \zeta_4$, which follows from the structure of the frames for **KB**.

□

B Supplementary Proof

In this appendix we explore the details of Claim 3.5 from the proof of Lemma 3.4, wherein we established the correspondence between satisfiability with respect to $\mathbf{Q}^{\#}\mathbf{K}$ and the existence of quasimodels.

Claim 3.5 *For all $w \in W$, $r \in \mathfrak{R}$, and $\psi \in \mathrm{sub}(\varphi)$,*

$$\mathfrak{M}, w \models^{\mathfrak{a}(x/r)} \psi \qquad \Longleftrightarrow \qquad \psi \in r(w). \tag{I.H.}$$

Proof. The claim follows by induction on the construction of ψ, with each of the five inductive cases detailed below:

- *Case* $\psi = P_i(x)$: This follows immediately from the definition of $P_i^{I(w)}$, since

$$\mathfrak{M}, w \models^{\mathfrak{a}(x/r)} P_i(x) \iff r \in P_i^{I(w)} \iff P_i(x) \in r(w).$$

- *Case* $\psi = \neg\xi$: We have that

$$\mathfrak{M}, w \models^{\mathfrak{a}(x/r)} \neg\xi \iff \mathfrak{M}, w \not\models^{\mathfrak{a}(x/r)} \xi \overset{\text{(I.H.)}}{\iff} \xi \notin r(w) \overset{\textbf{(tp1)}}{\iff} \neg\xi \in r(w).$$

- *Case* $\psi = (\xi_1 \wedge \xi_2)$: We have that

$$\begin{aligned} \mathfrak{M}, w \models^{\mathfrak{a}(x/r)} (\xi_1 \wedge \xi_2) &\iff \mathfrak{M}, w \not\models^{\mathfrak{a}(x/r)} \xi_1 \text{ and } \mathfrak{M}, w \not\models^{\mathfrak{a}(x/r)} \xi_2, \\ &\overset{\text{(I.H.)}}{\iff} \xi_1 \in r(w) \text{ and } \xi_2 \in r(w), \\ &\overset{\textbf{(tp2)}}{\iff} (\xi_1 \wedge \xi_2) \in r(w). \end{aligned}$$

- *Case* $\psi = \Diamond\xi$: We have that

$$\begin{aligned} \mathfrak{M}, w \models^{\mathfrak{a}(x/r)} \Diamond\xi &\iff w \prec v \text{ and } \mathfrak{M}, v \models^{\mathfrak{a}(x/r)} \xi, \quad \text{for some } v \in W, \\ &\overset{\text{(I.H.)}}{\iff} w \prec v \text{ and } \xi \in r(v), \quad \text{for some } v \in W, \\ &\iff \Diamond\alpha \in r(w) \quad \text{by } \textbf{(qm3)} \text{ and } \textbf{(qm4)}. \end{aligned}$$

- *Case* $\psi = \exists_{\leq c} x\ \xi$: We have that

$$\begin{aligned} \mathfrak{M}, w \models^{\mathfrak{a}(x/r)} (\exists_{\leq c} x\ \xi) &\overset{\text{(def)}}{\iff} \left| \{ s \in \mathfrak{R} \ : \ \mathfrak{M}, w \models^{\mathfrak{a}(x/s)} \xi \} \right| \ \leq \ c \\ &\overset{\text{(I.H.)}}{\iff} |\{ s \in \mathfrak{R} \ : \ \xi \in s(w) \}| \ \leq \ c \\ &\iff \sum_{t \in T_w(\xi)} |\{ s \in \mathfrak{R} \ : \ s(w) = t \}| \ \leq \ c \\ &\overset{\textbf{(qm5)}}{\iff} \sum_{t \in T_w(\xi)} \mu_w(t) \ \leq \ c \\ &\overset{\textbf{(qs3)}}{\iff} (\exists_{\leq c} x\ \xi) \in r(w). \end{aligned}$$

  The penultimate equivalence follows from the fact that each summand strictly less than $(C+1)$, since $c \leq C$. Hence, it follows from **(qm5)** that $\mu_w(t) = |\{ s \in \mathfrak{R} \ : \ s(w) = t \}|$, for all $t \in T_w(\xi)$.

Hence, it follow that $\mathfrak{M}, w \models^{\mathfrak{a}(x/r)} \psi$ if and only if $\psi \in r(w)$, for all $\psi \in \mathrm{sub}(\varphi)$, as required. □