Unification in modal logic Alt_1

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Abstract

Alt_1 is the least modal logic containing the formula $\Diamond x \rightarrow \Box x$. It is determined by the class of all deterministic frames. The unification problem in Alt_1 is to determine, given a formula $\phi(x_1, \ldots, x_\alpha)$, whether there exists formulas $\psi_1, \ldots, \psi_\alpha$ such that $\phi(\psi_1, \ldots, \psi_\alpha)$ is in Alt_1. In this paper, we show that the unification problem in Alt_1 is in $PSPACE$. We also show that there exists an Alt_1-unifiable formula that has no minimal complete set of unifiers. Finally, we study sub-Boolean variants of the unification problem in Alt_1.

Keywords: Modal logic Alt_1. Computability of unifiability. Unification type. Sub-Boolean variants.

1 Introduction

Modal logics are essential to the design of logical systems that capture elements of reasoning about knowledge, time, etc. There exists variants of these logics with one or several modalities, with or without the universal modality, etc. The logical problems addressed in their setting usually concern their axiomatizability, their decidability, etc. Other desirable properties which one should establish whenever possible concern, for example, the admissibility problem and the unifiability problem. About admissibility, an inference rule $\frac{\phi_1, \ldots, \phi_n}{\psi}$ is admissible in a modal logic L if for all instances $\frac{\phi'_1, \ldots, \phi'_n}{\psi'}$ of the inference rule, if $\phi'_1, \ldots, \phi'_n$ are in L then ψ' is in L too [18]. About unifiability, a formula ϕ is unifiable in a modal logic L if there exists an instance ϕ' of the formula such that ϕ' is in L [11]. When a modal logic L is axiomatically presented, its

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admissible inference rules can be added to its axiomatical presentation without changing the set of its theorems. As a result, in order to improve the efficiency of automated theorem provers for modal logics, methods for deciding the admissibility of inference rules can be used [8]. The unifiability problem is easily reducible to the admissibility problem, seeing that the formula ϕ is unifiable in L iff the inference rule $\frac{\phi}{\bot}$ is non-admissible in L. In some cases, when L's unification type is finitary, the admissibility problem is reducible to the unifiability problem. Therefore, in order to improve the efficiency of automated theorem provers for modal logics, methods for deciding the unifiability of formulas can be used as well.
Results about unification have been already obtained in many modal logics. Rybakov [17] demonstrated that unification in $S4$ is decidable. Wolter and Zakharyaschev [19] showed that unification is undecidable for $K4$ or K extended with the universal modality. The notion of projectivity has been introduced by Ghilardi [13] to determine the unification type, finitary, of $S4$ and $K4$. Jeřábek [16] established the unification type, nullary, of K. Within the context of description logics, checking subsumption of concepts is not sufficient and new inference capabilities are required. One of them, unification of concept terms, has been introduced by Baader and Narendran [6] for $\mathcal{FL}_0$. Baader and Küsters [4] established the $EXPTIME$-completeness of unification in $\mathcal{FL}_{reg}$ whereas Baader and Morawska [5] established the $NPTIME$-completeness of unification in $\mathcal{EL}$. Much remains to be done, seeing that the computability of unifiability and the unification types are unknown in multifarious modal logics. In this paper, we consider the unification problem in Alt_1. Its section-by-section breakdown is organized as follows. Section 2 defines the syntax, Section 3 introduces the semantics and Section 4 presents unification. In Section 5, useful Lemmas are proved. They are used in Section 6 to prove the soundness/completeness of a nondeterministic algorithm solving unification in polynomial space. In Section 7, it is shown that there exists a unifiable formula that has no minimal complete set of unifiers. In Section 8, we study sub-Boolean variants of unification.

2 Syntax

Let AF be a countable set of atomic formulas (denoted x, y, etc). The set F of all formulas (denoted ϕ, ψ, etc) is inductively defined as follows:

- $\phi ::= x \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi$.

We define the other Boolean constructs as usual. The formula $\Diamond\phi$ is obtained as an abbreviation:

- $\Diamond\phi ::= \neg\Box\neg\phi$.

The modal connective $\Box^k$ is inductively defined as follows for each $k \in \mathbb{N}$:

- $\Box^0\phi ::= \phi$,
- $\Box^{k+1}\phi ::= \Box\Box^k\phi$.

The modal connective $\Box^{<k}$ is inductively defined as follows for each $k \in \mathbb{N}$:

- $\Box^{<0}\phi ::= \top$,
- $\Box^{<k+1}\phi ::= \Box^{<k}\phi \wedge \Box^{k}\phi$.

We adopt the standard rules for omission of the parentheses. Let $deg(\phi)$ denote the degree of a formula ϕ and $var(\phi)$ its atom-set. We shall say that a formula ϕ is atom-free iff $var(\phi) = \emptyset$. Let AFF be the set of all atom-free formulas. In the sequel, we use $\phi(x_1, \ldots, x_\alpha)$ to denote a formula whose atomic formulas form a subset of $\{x_1, \ldots, x_\alpha\}$. A substitution is a function σ associating to each variable x a formula $\sigma(x)$. We shall say that a substitution σ is closed if for all variables x, $\sigma(x) \in AFF$. For all formulas $\phi(x_1, \ldots, x_\alpha)$, let $\sigma(\phi(x_1, \ldots, x_\alpha))$ be $\phi(\sigma(x_1), \ldots, \sigma(x_\alpha))$. The composition $\sigma \circ \tau$ of the substitutions σ and τ associates to each atomic formula x the formula $\tau(\sigma(x))$. Remark that for all substitutions σ, τ, if τ is closed then $\sigma \circ \tau$ is closed.

3 Semantics

Our modal language receives a relational semantics and a tuple semantics.

3.1 Relational semantics

A frame is a relational structure of the form $\mathcal{F} = (W, R)$ where W is a nonempty set of states (with typical members denoted s, t, etc) and R is a binary relation on W. A model based on a frame $\mathcal{F} = (W, R)$ is a relational structure of the form $\mathcal{M} = (W, R, V)$ where V is a function associating to each variable x a set $V(x)$ of states. We inductively define the truth of a formula ϕ in a model $\mathcal{M}$ at state s, in symbols $\mathcal{M}, s \models \phi$, as follows:

- $\mathcal{M}, s \models x$ iff $s \in V(x)$,
- $\mathcal{M}, s \not\models \bot$,
- $\mathcal{M}s \models \neg\phi$ iff $\mathcal{M}, s \not\models \phi$,
- $\mathcal{M}, s \models \phi \vee \psi$ iff either $\mathcal{M}, s \models \phi$, or $\mathcal{M}, s \models \psi$,
- $\mathcal{M}, s \models \Box\phi$ iff for all states $t \in W$, if sRt then $\mathcal{M}, t \models \phi$.

Obviously,

- $\mathcal{M}, s \models \Diamond\phi$ iff there exists a state $t \in W$ such that sRt and $\mathcal{M}, t \models \phi$,
- $\mathcal{M}, s \models \Box^{k}\phi$ iff for all states $t \in W$, if $sR^{k}t$ then $\mathcal{M}, t \models \phi$,
- $\mathcal{M}, s \models \Box^{<k}\phi$ iff for all states $t \in W$ and for all $i \in \mathbb{N}$, if $sR^{i}t$ and $i < k$ then $\mathcal{M}, t \models \phi$.

Let $\mathcal{C}$ be a class of frames. We shall say that a formula ϕ is $\mathcal{C}$-valid, in symbols $\mathcal{C} \models \phi$, if for all frames $\mathcal{F} = (W, R)$ in $\mathcal{C}$, for all models $\mathcal{M} = (W, R, V)$ based on $\mathcal{F}$ and for all states $s \in W$, $\mathcal{M}, s \models \phi$.

3.2 Tuple semantics

For all $n \in \mathbb{N}$, an n-valuation is an $(n+1)$-tuple $(U_0, \ldots, U_n)$ of subsets of AF. We inductively define the truth of a formula ϕ in an n-valuation $(U_0, \ldots, U_n)$, in symbols $(U_0,\ldots,U_n)\models\phi$, as follows:

- $(U_0,\ldots,U_n)\models x$ iff $x\in U_n$,
- $(U_0,\ldots,U_n)\not\models\bot$,
- $(U_0,\ldots,U_n)\models\neg\phi$ iff $(U_0,\ldots,U_n)\not\models\phi$,
- $(U_0,\ldots,U_n)\models\phi\vee\psi$ iff either $(U_0,\ldots,U_n)\models\phi$, or $(U_0,\ldots,U_n)\models\psi$,
- $(U_0,\ldots,U_n)\models\Box\phi$ iff if $n\geq 1$ then $(U_0,\ldots,U_{n-1})\models\phi$.

Obviously,

- $(U_0,\ldots,U_n)\models\Diamond\phi$ iff $n\geq 1$ and $(U_0,\ldots,U_{n-1})\models\phi$,
- $(U_0,\ldots,U_n)\models\Box^k\phi$ iff if $n\geq k$ then $(U_0,\ldots,U_{n-k})\models\phi$,
- $(U_0,\ldots,U_n)\models\Box^{<k}\phi$ iff for all $i\in\mathbb{N}$, if $n\geq i$ and $i<k$ then $(U_0,\ldots,U_{n-i})\models\phi$.

We shall say that a formula ϕ is n-tuple-valid, in symbols $\models_n\phi$, iff for all n-valuations $(U_0,\ldots,U_n)$, $(U_0,\ldots,U_n)\models\phi$.

3.3 Correspondence between the two semantics

In this paper, we will be only interested in the class $\mathcal{C}_{det}$ of all deterministic frames, i.e. frames $\mathcal{F}=(W,R)$ such that for all states $s,t,u\in W$, if sRt and sRu then $t=u$.

Proposition 3.1 *Let ϕ be a formula. The following conditions are equivalent:*

(i) $\mathcal{C}_{det}\models\phi$.

(ii) *For all $n\in\mathbb{N}$, $\models_n\phi$.*

When the conditions from Proposition 3.1 hold, we shall simply say that ϕ is valid, in symbols $\models\phi$.

4 Unification

We shall say that a formula $\phi(x_1,\ldots,x_\alpha)$ is unifiable iff there exists $\psi_1,\ldots,\psi_\alpha\in F$ such that $\models\phi(\psi_1,\ldots,\psi_\alpha)$. In that case, the substitution σ defined by $\sigma(x_1)=\psi_1$, …, $\sigma(x_\alpha)=\psi_\alpha$ is called unifier of ϕ. For instance, the formula $\phi=\Box x\vee\Box y$ is unifiable. The substitution σ defined by $\sigma(x)=z$ and $\sigma(y)=\neg z$ is a unifier of ϕ. Remark that if a formula possesses a unifier then it possesses a closed unifier. This follows from the fact that for all unifiers σ of a formula ϕ and for all closed substitutions τ, $\sigma\circ\tau$ is a closed unifier of ϕ. The unification problem is the decision problem defined as follows:

- given a formula $\phi(x_1,\ldots,x_\alpha)$, determine whether $\phi(x_1,\ldots,x_\alpha)$ is unifiable.

We shall say that a substitution σ is equivalent to a substitution τ, in symbols $\sigma\simeq\tau$, if for all variables x, $\models\sigma(x)\leftrightarrow\tau(x)$. We shall say that a substitution σ is more general than a substitution τ, in symbols $\sigma\preceq\tau$, if there exists a substitution υ such that $\sigma\circ\upsilon\simeq\tau$. We shall say that a set Σ of unifiers of a unifiable formula ϕ is complete if for all unifiers σ of ϕ, there exists a unifier τ of ϕ in Σ such that $\tau\preceq\sigma$. An important question is the following: when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is "yes", how large is this set? We shall say that a unifiable formula

- ϕ is unitary if there exists a minimal complete set of unifiers of ϕ with cardinality 1,
- ϕ is finitary if there exists a finite minimal complete set of unifiers of ϕ but there exists no with cardinality 1,
- ϕ is infinitary if there exists an infinite minimal complete set of unifiers of ϕ but there exists no finite one,
- ϕ is nullary if there exists no minimal complete set of unifiers of ϕ.

For instance, the formula x is unitary: the substitution σ defined by $\sigma(x) = \top$ constitutes a minimal complete set of unifiers of it. We do not know whether there exists finitary, or infinitary formulas. We will show in Section 7 that the formula $x \rightarrow \Box x$ is nullary.

5 Unification problem: lemmas

Let $\psi(x)$ be an arbitrary formula with at most one atomic formula.

Lemma 5.1 *For all $k \in \mathbb{N}$, the following conditions are equivalent:*

(i) *$\psi(x)$ is unifiable;*

(ii) *there exists $\phi \in AFF$ such that $\models \psi(\phi)$;*

(iii) *there exists $\phi \in AFF$ such that $\models \Box^k\bot \rightarrow \psi(\phi)$ and $\models \Diamond^k\top \rightarrow \psi(\phi)$.*

Remark that Lemma 5.1 still holds when one considers a formula $\psi(x_1, \ldots, x_\alpha)$ with more than one atomic formula. In this case, simply replace the "there exists ϕ ..." by "there exists $\phi_1, \ldots, \phi_\alpha$ ...". Concerning the remainder of this Section and Section 6, the same remark is on as well. Hence, without loss of generality, we will always consider in the remainder of this Section and in Section 6 that ψ is a formula with at most one atomic formula. In this case, for all $n \in \mathbb{N}$, an n-valuation is comparable to an $(n+1)$-tuple of bits. Let $k \in \mathbb{N}$ be such that $deg(\psi(x)) \leq k$. For all $\phi \in AFF$ and for all $n \in \mathbb{N}$, if $k \leq n$ then let $V_k(\phi, n, i) =$ "if $\models_{n-k+i} \phi$ then 1 else 0" for each $i \in \mathbb{N}$ such that $i \leq k$.

Lemma 5.2 *For all $\phi \in AFF$ and for all $n \in \mathbb{N}$, if $k \leq n$ then the following conditions are equivalent:*

(i) *$\models_n \psi(\phi)$;*

(ii) *$(V_k(\phi, n, 0), \ldots, V_k(\phi, n, k)) \models \psi(x)$.*

Lemma 5.3 *For all $\phi \in AFF$, the following conditions are equivalent:*

(i) *$\models \Diamond^k\top \rightarrow \psi(\phi)$;*

(ii) *for all $n \in \mathbb{N}$, if $k \leq n$ then $(V_k(\phi, n, 0), \ldots, V_k(\phi, n, k)) \models \psi(x)$.*

For all $\phi \in AFF$ and for all $n \in \mathbb{N}$, if $k \leq n$ then let $\boldsymbol{V}_k(\phi, n) = (V_k(\phi, n, 0), \ldots, V_k(\phi, n, k))$. For all $\phi \in AFF$, let $f_k(\phi) = \{\boldsymbol{V}_k(\phi, n) \colon n \in$

$\mathbb{N}$ is such that $k \leq n\}$. The atom-free formulas ϕ' and ϕ'' are said to be k-equivalent, in symbols $\phi' \equiv_k \phi''$, iff $f_k(\phi') = f_k(\phi'')$.

Proposition 5.4 *$\equiv_k$ is an equivalence relation on AFF possessing finitely many equivalence classes.*

Proof. By definitions of $\equiv_k$ and f_k, knowing that for all $\phi \in AFF$, $f_k(\phi)$ is a nonempty set of $(k+1)$-tuples of bits. ☐

Lemma 5.5 *For all $\phi', \phi'' \in AFF$, if $\phi' \equiv_k \phi''$ then the following conditions are equivalent:*

(i) $\models \Diamond^k \top \to \psi(\phi')$;

(ii) $\models \Diamond^k \top \to \psi(\phi'')$.

For all $\phi \in AFF$ and for all $n \in \mathbb{N}$, let $\boldsymbol{a}_k(\phi, n) = \boldsymbol{V}_k(\phi, n \cdot (k+1) + k)$. For all $\phi \in AFF$, let $g_k(\phi) = \{(\boldsymbol{a}_k(\phi, n), \boldsymbol{a}_k(\phi, n+1))\colon n \in \mathbb{N}\}$. We shall say that the atom-free formulas ϕ' and ϕ'' are k-congruent, in symbols $\phi' \cong_k \phi''$, iff $g_k(\phi') = g_k(\phi'')$.

Proposition 5.6 *$\cong_k$ is an equivalence relation on AFF possessing finitely many equivalence classes.*

Proof. By definitions of $\cong_k$ and g_k, knowing that for all $\phi \in AFF$, $g_k(\phi)$ is a nonempty set of pairs of $(k+1)$-tuples of bits. ☐

Proposition 5.7 *For all $\phi', \phi'' \in AFF$, if $\phi' \cong_k \phi''$ then $\phi' \equiv_k \phi''$.*

Proof. Let $\phi', \phi'' \in AFF$. Suppose $\phi' \cong_k \phi''$ and $\phi' \not\equiv_k \phi''$. Hence, $g_k(\phi') = g_k(\phi'')$ and $f_k(\phi') \neq f_k(\phi'')$. Thus, either there exists $n' \in \mathbb{N}$ such that $k \leq n'$ and $\boldsymbol{V}_k(\phi', n') \not\in f_k(\phi'')$, or there exists $n'' \in \mathbb{N}$ such that $k \leq n''$ and $\boldsymbol{V}_k(\phi'', n'') \not\in f_k(\phi')$. Without loss of generality, assume there exists $n' \in \mathbb{N}$ such that $k \leq n'$ and $\boldsymbol{V}_k(\phi', n') \not\in f_k(\phi'')$. By the division algorithm, let $m, l \in \mathbb{N}$ be such that $n' = m \cdot (k+1) + l$ and $l < k+1$.
Case $m = 0$. Since $k \leq n'$, $n' = m \cdot (k+1) + l$ and $l < k+1$, therefore $n' = k$. Hence, $\boldsymbol{V}_k(\phi', n') = \boldsymbol{a}_k(\phi', 0)$. Since $g_k(\phi') = g_k(\phi'')$, therefore let $n'' \in \mathbb{N}$ be such that $(\boldsymbol{a}_k(\phi', 0), \boldsymbol{a}_k(\phi', 1)) = (\boldsymbol{a}_k(\phi'', n''), \boldsymbol{a}_k(\phi'', n''+1))$. Since $\boldsymbol{V}_k(\phi', n') = \boldsymbol{a}_k(\phi', 0)$, therefore $\boldsymbol{V}_k(\phi', n') = \boldsymbol{V}_k(\phi'', n'' \cdot (k+1) + k)$.
Case $m \neq 0$. Since $g_k(\phi') = g_k(\phi'')$, therefore let $n'' \in \mathbb{N}$ be such that $(\boldsymbol{a}_k(\phi', m-1), \boldsymbol{a}_k(\phi', m)) = (\boldsymbol{a}_k(\phi'', n''), \boldsymbol{a}_k(\phi'', n''+1))$. Hence, $V_k(\phi', (m-1) \cdot (k+1) + k, i) = V_k(\phi'', n'' \cdot (k+1) + k, i)$ and $V_k(\phi', m \cdot (k+1) + k, i) = V_k(\phi'', (n''+1) \cdot (k+1) + k, i)$ for each $i \in \mathbb{N}$ such that $i \leq k$. Since either $n' = m \cdot (k+1) + l$ and $i \leq k - (l+1)$ and $V_k(\phi', m \cdot (k+1) + l, i) = V_k(\phi', (m-1) \cdot (k+1) + k, i + (l+1))$, or $k - l \leq i$ and $V_k(\phi', m \cdot (k+1) + l, i) = V_k(\phi', m \cdot (k+1) + k, i - (k-l))$ for each $i \in \mathbb{N}$ such that $i \leq k$, therefore either $i \leq k - (l+1)$ and $V_k(\phi', n', i) = V_k(\phi'', n'' \cdot (k+1) + k, i + (l+1))$, or $k - l \leq i$ and $V_k(\phi', n', i) = V_k(\phi'', (n''+1) \cdot (k+1) + k, i - (k-l))$ for each $i \in \mathbb{N}$ such that $i \leq k$. Thus, $V_k(\phi', n', i) = V_k(\phi'', (n''+1) \cdot (k+1) + l, i)$ for each $i \in \mathbb{N}$ such that $i \leq k$. Consequently, $\boldsymbol{V}_k(\phi', n') = \boldsymbol{V}_k(\phi'', (n''+1) \cdot (k+1) + l)$.
In both cases, $\boldsymbol{V}_k(\phi', n') \in f_k(\phi'')$: a contradiction. ☐

Lemma 5.8 *For all $\phi', \phi'' \in AFF$, if $\phi' \cong_k \phi''$ then the following conditions are equivalent:*

(i) $\models \Diamond^k \top \to \psi(\phi')$;

(ii) $\models \Diamond^k \top \to \psi(\phi'')$.

We shall say that a nonempty set B of pairs of $(k+1)$-tuples of bits is modally definable iff there exists $\phi \in AFF$ such that $B = g_k(\phi)$. For all nonempty sets B of pairs of $(k+1)$-tuples of bits, let $\triangleright_B$ be the domino relation on B defined as follows:

- $(b'_1, b''_1) \triangleright_B (b'_2, b''_2)$ iff $b''_1 = b'_2$.

We shall say that a path in the directed graph $(B, \triangleright_B)$ is weakly Hamiltonian iff it visits each vertex at least once. Let $\mathbf{1}_{k+1}$ be the $(k+1)$-tuple of 1 and $\mathbf{0}_{k+1}$ be the $(k+1)$-tuple of 0.

Proposition 5.9 *For all nonempty sets B of pairs of $(k+1)$-tuples of bits, the following conditions are equivalent:*

(i) *B is modally definable;*

(ii) *the directed graph $(B, \triangleright_B)$ contains a weakly Hamiltonian path either ending with $(\mathbf{1}_{k+1}, \mathbf{1}_{k+1})$, or ending with $(\mathbf{0}_{k+1}, \mathbf{0}_{k+1})$.*

Proof. Let B be a nonempty set of pairs of $(k+1)$-tuples of bits.
If. Suppose the directed graph $(B, \triangleright_B)$ contains a weakly Hamiltonian path either ending with $(\mathbf{1}_{k+1}, \mathbf{1}_{k+1})$, or ending with $(\mathbf{0}_{k+1}, \mathbf{0}_{k+1})$. Let $s \in \mathbb{N}$ and $(b'_0, b''_0), \ldots, (b'_s, b''_s) \in B$ be such that $((b'_0, b''_0), \ldots, (b'_s, b''_s))$ is a weakly Hamiltonian path either ending with $(\mathbf{1}_{k+1}, \mathbf{1}_{k+1})$, or ending with $(\mathbf{0}_{k+1}, \mathbf{0}_{k+1})$. Let $(\beta_0, \ldots, \beta_{s\cdot(k+1)+k})$ be the sequence of bits determined by the sequence $(b'_0, , \ldots, b'_s)$ of $(k+1)$-tuples of bits.
Case $(b'_s, b''_s) = (\mathbf{1}_{k+1}, \mathbf{1}_{k+1})$**.** Let $\phi = \bigvee\{\Diamond^i\Box\bot\colon i \in \mathbb{N}$ is such that $i < s\cdot(k+1)$ and $\beta_i = 1\} \vee \Diamond^{s\cdot(k+1)}\top$.
Case $(b'_s, b''_s) = (\mathbf{0}_{k+1}, \mathbf{0}_{k+1})$**.** Let $\phi = \bigvee\{\Diamond^i\Box\bot\colon i \in \mathbb{N}$ is such that $i < s\cdot(k+1)$ and $\beta_i = 1\}$.
In both cases, the reader may easily verify that for all $n \in \mathbb{N}$, if $n \leq s$ then $V_k(\phi, n\cdot(k+1)+k, i) = \beta_{n\cdot(k+1)+i}$ for each $i \in \mathbb{N}$ such that $i \leq k$. Hence, for all $n \in \mathbb{N}$, if $n \leq s$ then $\boldsymbol{V}_k(\phi, n\cdot(k+1)+k) = b'_n$. Thus, for all $n \in \mathbb{N}$, if $n \leq s$ then $(\boldsymbol{a}_k(\phi, n), \boldsymbol{a}_k(\phi, n+1)) = (b'_n, b''_n)$. Hence, $B = g_k(\phi)$.
Only if. Suppose B is modally definable. Let $\phi \in AFF$ be such that $B = g_k(\phi)$. Let $n_0 \in \mathbb{N}$ be such that either for all $n \in \mathbb{N}$, if $n_0 \leq n$ then $\boldsymbol{a}_k(\phi, n) = \mathbf{1}_{k+1}$, or for all $n \in \mathbb{N}$, if $n_0 \leq n$ then $\boldsymbol{a}_k(\phi, n) = \mathbf{0}_{k+1}$. Thus, $((\boldsymbol{a}_k(\phi, 0), \boldsymbol{a}_k(\phi, 1)), \ldots, (\boldsymbol{a}_k(\phi, n_0), \boldsymbol{a}_k(\phi, n_0+1)))$ is a weakly Hamiltonian path either ending with $(\mathbf{1}_{k+1}, \mathbf{1}_{k+1})$, or ending with $(\mathbf{0}_{k+1}, \mathbf{0}_{k+1})$. $\Box$

6 Unification problem: algorithm

As in Section 5, let $\psi(x)$ be an arbitrary formula with at most one atomic formula and $k \in \mathbb{N}$ be such that $deg(\psi(x)) \leq k$. We shall say that an infinite sequence $(\beta_0, \beta_1, \ldots)$ of bits respects $\psi(x)$ iff the following conditions hold:

- for all $i \in \mathbb{N}$, if $i \leq k$ then $(\beta_0, \ldots, \beta_i) \models \psi(x)$,
- for all $i \in \mathbb{N}$, $(\beta_{i+1}, \ldots, \beta_{i+k+1}) \models \psi(x)$.

Using the above results, $\psi(x)$ is unifiable iff there exists a modally definable set B of pairs of $(k+1)$-tuples of bits from which, by means of its domino relation, an infinite sequence of bits respecting $\psi(x)$ and either ending with 1s, or ending with 0s can be constructed. Hence, in order to determine whether $\psi(x)$ is unifiable, it suffices to consider the following procedure:

```
procedure UNI(ψ(x))
begin
k := deg(ψ(x))
guess a tuple (b(0), ..., b(k)) of bits of size k + 1
bool := ⊤
i := 0
while bool ∧ i ≤ k do
    begin
    bool := MC(b(0), ..., b(i), ψ(x))
    i := i + 1
    end
if ¬bool then reject
while (b(0), ..., b(k)) ≠ 0_{k+1} ∧ (b(0), ..., b(k)) ≠ 1_{k+1} do
    begin
    guess a tuple (b(k + 1), ..., b(2k + 1)) of bits of size k + 1
    bool := ⊤
    i := 0
    while bool ∧ i ≤ k do
        begin
        bool := MC(b(i + 1), ..., b(i + k + 1), ψ(x))
        i := i + 1
        end
    if ¬bool then reject
    (b(0), ..., b(k)) := (b(k + 1), ..., b(2k + 1))
    end
accept
end
```

The function $MC(\cdot)$ takes as input a tuple $(b(i), \ldots, b(i+j))$ of bits and a formula $\psi(x)$ and returns the Boolean value

- $MC(b(i), \ldots, b(i+j), \psi(x)) =$ "if $(b(i), \ldots, b(i+j)) \models \psi(x)$ then $\top$ else $\bot$".

It can be implemented as a deterministic Turing machine working in polynomial time. The procedure $UNI(\cdot)$ takes as input a formula $\psi(x)$ and accepts it iff, when $k = deg(\psi(x))$, there exists a modally definable set B of pairs of $(k+1)$-tuples of bits from which, by means of its domino relation, an infinite sequence of bits respecting $\psi(x)$ and either ending with 1s, or ending with 0s can be constructed. By Proposition 5.9, the procedure $UNI(\cdot)$ accepts its input $\psi(x)$ iff $\psi(x)$ is unifiable. It can be implemented as a nondeterministic Turing machine working in polynomial space. Hence, the unification problem is in $NPSPACE$. Since $NPSPACE = PSPACE$, therefore

Proposition 6.1 *The unification problem is in $PSPACE$.*

Still, we do not know whether the unification problem is $PSPACE$-hard.

7 Unification type

Following the line of reasoning suggested by Jeřábek [16], we consider the formula $\phi(x) = x \to \Box x$. We also consider the substitution $\sigma_\top$ defined by $\sigma_\top(x) = \top$ and for all $k \in \mathbb{N}$, the substitution σ_k defined by $\sigma_k(x) = \Box^{<k} x \wedge \Box^k \bot$.

Lemma 7.1 • *$\sigma_\top$ is a unifier of $\phi(x)$,*

• *for all $k \in \mathbb{N}$, σ_k is a unifier of $\phi(x)$.*

Lemma 7.2 *Let $k, l \in \mathbb{N}$. If $k \leq l$ then $\sigma_l \preceq \sigma_k$.*

Lemma 7.3 *Let $k, l \in \mathbb{N}$. If $k < l$ then $\sigma_k \not\preceq \sigma_l$.*

Proposition 7.4 *Let σ be a substitution. The following conditions are equivalent:*

(i) $\sigma_\top \circ \sigma \simeq \sigma$.

(ii) $\sigma_\top \preceq \sigma$.

(iii) $\models \sigma(x)$.

Proof. (i⇒ii) By definition of $\simeq$ and $\preceq$.
(ii⇒iii) Suppose $\sigma_\top \preceq \sigma$. Let τ be a substitution such that $\sigma_\top \circ \tau \simeq \sigma$. Thus, $\models \tau(\sigma_\top(x)) \leftrightarrow \sigma(x)$. Hence, $\models \top \leftrightarrow \sigma(x)$. Consequently, $\models \sigma(x)$.
(iii⇒i) Suppose $\models \sigma(x)$. Hence, $\models \top \leftrightarrow \sigma(x)$. Thus, $\models \sigma(\sigma_\top(x)) \leftrightarrow \sigma(x)$. Consequently, $\sigma_\top \circ \sigma \simeq \sigma$. $\Box$

Proposition 7.5 *Let σ be a unifier of $\phi(x)$ and $k \in \mathbb{N}$. The following conditions are equivalent:*

(i) $\sigma_k \circ \sigma \simeq \sigma$.

(ii) $\sigma_k \preceq \sigma$.

(iii) $\models \sigma(x) \to \Box^k \bot$.

Proof. (i⇒ii) By definition of $\simeq$ and $\preceq$.
(ii⇒iii) Suppose $\sigma_k \preceq \sigma$. Let τ be a substitution such that $\sigma_k \circ \tau \simeq \sigma$. Thus, $\models \tau(\sigma_k(x)) \leftrightarrow \sigma(x)$. Hence, $\models \Box^{<k}\tau(x) \wedge \Box^k \bot \leftrightarrow \sigma(x)$. Consequently, $\models \sigma(x) \to \Box^k \bot$.
(iii⇒i) Suppose $\models \sigma(x) \to \Box^k \bot$. Obviously, $\models \Box^{<k}\sigma(x) \wedge \Box^k \bot \leftrightarrow \sigma(\sigma_k(x))$. Hence, $\models \sigma(\sigma_k(x)) \to \sigma(x)$. Since σ is a unifier of $\phi(x)$, therefore $\models \sigma(x) \to \Box\sigma(x)$. Thus, $\models \sigma(x) \to \Box^{<k}\sigma(x)$. Since $\models \sigma(x) \to \Box^k \bot$, therefore $\models \sigma(x) \to \Box^{<k}\sigma(x) \wedge \Box^k \bot$. Since $\models \Box^{<k}\sigma(x) \wedge \Box^k \bot \leftrightarrow \sigma(\sigma_k(x))$, therefore $\models \sigma(x) \to \sigma(\sigma_k(x))$. Since $\models \sigma(\sigma_k(x)) \to \sigma(x)$, therefore $\models \sigma(\sigma_k(x)) \leftrightarrow \sigma(x)$. Consequently, $\sigma_k \circ \sigma \simeq \sigma$. $\Box$

Proposition 7.6 *Let σ be a unifier of $\phi(x)$ and $k \in \mathbb{N}$ be such that $\deg(\sigma(x)) \leq k$. One of the following conditions holds:*

(i) $\models \sigma(x)$.

(ii) $\models \sigma(x) \to \Box^k \bot$.

Proof. Suppose $\not\models \sigma(x)$ and $\not\models \sigma(x) \to \Box^k \bot$. Let $m, n \in \mathbb{N}$, $(U_0, \ldots, U_m)$ be an m-valuation such that $(U_0, \ldots, U_m) \not\models \sigma(x)$ and $(V_0, \ldots, V_n)$ be an n-valuation such that $(V_0, \ldots, V_n) \not\models \sigma(x) \to \Box^k \bot$. Hence, $(V_0, \ldots, V_n) \models \sigma(x)$ and $(V_0, \ldots, V_n) \not\models \Box^k \bot$. Thus, $n \geq k$. Since $\deg(\sigma(x)) \leq k$, therefore $n \geq \deg(\sigma(x))$. Since $(V_0, \ldots, V_n) \models \sigma(x)$, therefore $(U_0, \ldots, U_m, V_0, \ldots, V_n) \models \sigma(x)$. Since σ is a unifier of $\phi(x)$, therefore $\models \sigma(x) \to \Box\sigma(x)$. Consequently, $\models \sigma(x) \to \Box^{n+1}\sigma(x)$. Since $(U_0, \ldots, U_m, V_0, \ldots, V_n) \models \sigma(x)$, therefore $(U_0, \ldots, U_m, V_0, \ldots, V_n) \models \Box^{n+1}\sigma(x)$. Hence, $(U_0, \ldots, U_m) \models \sigma(x)$: a contradiction. $\Box$

Proposition 7.7 *$\phi(x)$ is nullary.*

Proof. Let $\Sigma = \{\sigma_\top\} \cup \{\sigma_k : k \in \mathbb{N}\}$. By Lemma 7.1 and Propositions 7.4–7.6, Σ is a complete set of unifiers of $\phi(x)$. Suppose there exists a minimal complete set of unifiers of $\phi(x)$. Let Γ be a minimal complete set of unifiers of $\phi(x)$. Let $\gamma \in \Gamma$ be such that $\gamma \preceq \sigma_0$. Since Σ is a complete set of unifiers of $\phi(x)$, therefore let $\sigma \in \Sigma$ be such that $\sigma \preceq \gamma$. Now, we consider the following 2 cases.
Case $\sigma = \sigma_\top$. Since $\gamma \preceq \sigma_0$, therefore $\sigma \preceq \sigma_0$. Let υ be a substitution such that $\sigma \circ \upsilon \simeq \sigma_0$. Hence, $\models \upsilon(\sigma(x)) \leftrightarrow \sigma_0(x)$. Thus, $\models \top \leftrightarrow \bot$: a contradiction.
Case $\sigma = \sigma_k$ for some $k \in \mathbb{N}$. By Lemma 7.3, $\sigma \not\preceq \sigma_{k+1}$. Let $\gamma' \in \Gamma$ be such that $\gamma' \preceq \sigma_{k+1}$. By Lemma 7.2, since $\sigma \preceq \gamma$, therefore $\gamma' \preceq \gamma$. Since Γ is a minimal complete set of unifiers of $\phi(x)$, therefore $\gamma' = \gamma$. Since $\gamma' \preceq \sigma_{k+1}$ and $\sigma \preceq \gamma$, therefore $\sigma \preceq \sigma_{k+1}$: a contradiction. $\Box$

8 Sub-Boolean variants

In this section, we study sub-Boolean variants of the unification problem.

8.1 $\{\Box, \top, \wedge\}$-fragment

In the $\{\Box, \top, \wedge\}$-fragment, formulas are defined as follows:

- $\phi ::= x \mid \top \mid (\phi \wedge \psi) \mid \Box\phi$.

Lemma 8.1 *Let σ, τ, υ be substitutions such that for all variables x, $var(\sigma(x)) \cap var(\tau(x)) = \emptyset$ and $\models \upsilon(x) \leftrightarrow \sigma(x) \wedge \tau(x)$. Then $\upsilon \preceq \sigma$ and $\upsilon \preceq \tau$.*

Lemma 8.2 *Let ϕ be a formula and σ, τ, υ be substitutions. If for all variables x, $\models \upsilon(x) \leftrightarrow \sigma(x) \wedge \tau(x)$ then $\models \upsilon(\phi) \leftrightarrow \sigma(\phi) \wedge \tau(\phi)$.*

The $\Box$-integer-set of a variable x with respect to a formula ϕ, in symbols $is_\Box(x, \phi)$, is inductively defined as follows:

- $is_\Box(x, y) = \{0\}$ if $x = y$,

- $is_\Box(x, y) = \emptyset$ if $x \neq y$,
- $is_\Box(x, \top) = \emptyset$,
- $is_\Box(x, \phi \wedge \psi) = is_\Box(x, \phi) \cup is_\Box(x, \psi)$,
- $is_\Box(x, \Box\phi) = \{i + 1 : \ i \in is_\Box(x, \phi)\}$.

For instance, $is_\Box(x, y \wedge \Box\Box z) = \emptyset$, $is_\Box(y, y \wedge \Box\Box z) = \{0\}$ and $is_\Box(z, y \wedge \Box\Box z) = \{2\}$. Let ϕ be an arbitrary formula.

Lemma 8.3 *For all $x \in AF$, $is_\Box(x, \phi)$ is a finite set such that $is_\Box(x, \phi) \neq \emptyset$ iff $x \in var(\phi)$. Moreover, $\models \phi \leftrightarrow \bigwedge\{\Box^i x : \ x \in AF \ \& \ i \in is_\Box(x, \phi)\}$.*

Lemma 8.4 *Let σ be a substitution and x be a variable. If $\models \sigma(\phi) \leftrightarrow x$ then there exists a variable y such that $0 \in is_\Box(y, \phi)$ and $\models \sigma(y) \leftrightarrow x$.*

Lemma 8.5 *Let σ be a substitution, $i \geq 0$ and x, y be variables. If $i \in is_\Box(x, \phi)$ and $\models \sigma(x) \leftrightarrow y$ then $i \in is_\Box(y, \sigma(\phi))$.*

In the $\{\Box, \top, \wedge\}$-fragment, unification problems are finite sets of pairs of formulas. We shall say that a finite set $S = \{(\phi_1, \psi_1), \ldots, (\phi_n, \psi_n)\}$ of pairs of formulas is unifiable iff there exists a substitution σ such that $\models \sigma(\phi_1) \leftrightarrow \sigma(\psi_1)$, $\ldots$, $\models \sigma(\phi_n) \leftrightarrow \sigma(\psi_n)$. In that case, σ is called a unifier of S. Of course, now, substitutions are functions associating to each variable a formula in the $\{\Box, \top, \wedge\}$-fragment. Obviously, if a finite set of pairs of formulas possesses a unifier then it possesses a closed unifier. Moreover, by Lemma 8.3, every atom-free formula is equivalent to $\top$. As a result,

Proposition 8.6 *Every finite set of pairs of formulas possesses a unifier.*

The simplicity of unification problems in the $\{\Box, \top, \wedge\}$-fragment does not entail that every finite set of pairs of formulas possesses a minimal complete set of unifiers. Following the line of reasoning suggested by Baader [2], we consider the formulas $\phi(x, y, z) = \Box x \wedge \Box y$ and $\psi(x, y, z) = y \wedge \Box\Box z$. We also consider for all $k \in \mathbb{N}$, the substitution σ_k defined by $\sigma_k(x) = t_k$, $\sigma_k(y) = \Box\Box^{<k+1} t_k$ and $\sigma_k(z) = \Box^k t_k$. We will assume that for all $k, l \in \mathbb{N}$, $k \neq l$, the variables t_k and t_l are distinct.

Lemma 8.7 *For all $k \in \mathbb{N}$, σ_k is a unifier of $\{(\phi(x, y, z), \psi(x, y, z))\}$.*

For all $k \in \mathbb{N}$, we consider

- the substitution γ_k inductively defined as follows:
  - $\gamma_0 = \sigma_0$,
  - γ_{k+1} is the substitution defined by $\gamma_{k+1}(x) = \gamma_k(x) \wedge \sigma_{k+1}(x)$, $\gamma_{k+1}(y) = \gamma_k(y) \wedge \sigma_{k+1}(y)$ and $\gamma_{k+1}(z) = \gamma_k(z) \wedge \sigma_{k+1}(z)$.

Lemma 8.8 *For all $k \in \mathbb{N}$, γ_k is a unifier of $\{(\phi(x, y, z), \psi(x, y, z))\}$.*

Lemma 8.9 *Let $k, l \in \mathbb{N}$. If $k \leq l$ then $\gamma_l \preceq \sigma_k$.*

Lemma 8.10 *Let $k, l \in \mathbb{N}$. If $k \leq l$ then $\gamma_l \preceq \gamma_k$.*

Proposition 8.11 *Let σ be a unifier of $\{(\phi(x, y, z), \psi(x, y, z))\}$ and $k \in \mathbb{N}$. If $\sigma \preceq \sigma_k$ then there exists a variable u such that $k \in is_\Box(u, \sigma(z))$.*

Proof. Suppose $\sigma \preceq \sigma_k$. Let τ be a substitution such that $\sigma \circ \tau \simeq \sigma_k$. Thus, $\models \tau(\sigma(x)) \leftrightarrow \sigma_k(x)$. Hence, $\models \tau(\sigma(x)) \leftrightarrow t_k$. By Lemma 8.4, let u be a variable such that $0 \in is_{\Box}(u, \sigma(x))$ and $\models \tau(u) \leftrightarrow t_k$. Since σ is a unifier of $\{(\phi(x, y, z), \psi(x, y, z))\}$, therefore $\models \Box\sigma(x) \wedge \Box\sigma(y) \leftrightarrow \sigma(y) \wedge \Box\Box\sigma(z)$. Since $0 \in is_{\Box}(u, \sigma(x))$, therefore $1 \in is_{\Box}(u, \sigma(y))$.
Claim: Let $i \geq 1$. If $i \in is_{\Box}(u, \sigma(y))$ and $i - 1 \notin is_{\Box}(u, \sigma(z))$ then $i + 1 \in is_{\Box}(u, \sigma(y))$.
Proof of the Claim: Suppose $i \in is_{\Box}(u, \sigma(y))$ and $i-1 \notin is_{\Box}(u, \sigma(z))$. Since $\models \Box\sigma(x) \wedge \Box\sigma(y) \leftrightarrow \sigma(y) \wedge \Box\Box\sigma(z)$, therefore $i + 1 \in is_{\Box}(u, \sigma(y))$.
By the above Claim, let $i \geq 1$ be such that $i-1 \in is_{\Box}(u, \sigma(z))$. By Lemma 8.5, since $\models \tau(u) \leftrightarrow t_k$, therefore $i-1 \in is_{\Box}(t_k, \tau(\sigma(z)))$. Since $\sigma \circ \tau \simeq \sigma_k$, therefore $\models \tau(\sigma(z)) \leftrightarrow \sigma_k(z)$. Hence, $\models \tau(\sigma(z)) \leftrightarrow \Box^k t_k$. Since $i - 1 \in is_{\Box}(t_k, \tau(\sigma(z)))$, therefore $i - 1 = k$. Since $i - 1 \in is_{\Box}(u, \sigma(z))$, therefore $k \in is_{\Box}(u, \sigma(z))$. $\Box$

Lemma 8.12 *Let σ be substitution. If σ is a unifier of $\{(\phi(x, y, z), \psi(x, y, z))\}$ then $\sigma \preceq \gamma_k$ for at most finitely many $k \in \mathbb{N}$.*

Proposition 8.13 *There exists no minimal complete set of unifiers of $\{(\phi(x, y, z), \psi(x, y, z))\}$.*

Proof. Let Δ be a minimal complete set of unifiers of $\{(\phi(x, y, z), \psi(x, y, z))\}$. Let $\delta \in \Delta$ be such that $\delta \preceq \sigma_0$. Hence, $\delta \preceq \gamma_0$. By Lemmas 8.10 and 8.12, let $k \in \mathbb{N}$ be such that $\delta \preceq \gamma_k$ and $\delta \not\preceq \gamma_{k+1}$. Without loss of generality, we can assume that $var(\delta(x)) \cap var(\gamma_{k+1}(x)) = \emptyset$, $var(\delta(y)) \cap var(\gamma_{k+1}(y)) = \emptyset$ and $var(\delta(z)) \cap var(\gamma_{k+1}(z)) = \emptyset$. Let ϵ be the substitution defined by $\epsilon(x) = \delta(x) \wedge \gamma_{k+1}(x)$, $\epsilon(y) = \delta(y) \wedge \gamma_{k+1}(y)$ and $\epsilon(z) = \delta(z) \wedge \gamma_{k+1}(z)$. By Lemmas 8.1, 8.2 and 8.8, ϵ is a unifier of $\{(\phi(x, y, z), \psi(x, y, z))\}$, $\epsilon \preceq \delta$ and $\epsilon \preceq \gamma_{k+1}$. Since Δ is a minimal complete set of unifiers of $\{(\phi(x, y, z), \psi(x, y, z))\}$, therefore let $\delta' \in \Delta$ be such that $\delta' \preceq \epsilon$. Since $\epsilon \preceq \delta$, therefore $\delta' \preceq \delta$. Since Δ is a minimal complete set of unifiers of $\{(\phi(x, y, z), \psi(x, y, z))\}$, therefore $\delta' = \delta$. Since $\epsilon \preceq \gamma_{k+1}$ and $\delta' \preceq \epsilon$, therefore $\delta \preceq \gamma_{k+1}$: a contradiction. $\Box$

8.2 $\{\Diamond, \top, \wedge\}$-fragment

In the $\{\Diamond, \top, \wedge\}$-fragment, formulas are defined as follows:

- $\phi ::= x \mid \top \mid (\phi \wedge \psi) \mid \Diamond\phi$.

The $\Diamond$-integer-set of a variable x with respect to a formula ϕ, in symbols $is_{\Diamond}(x, \phi)$, is inductively defined as has been defined the $\Box$-integer-set of x with respect to ϕ. Let ϕ be an arbitrary formula.

Lemma 8.14 *For all $x \in AF$, $is_{\Diamond}(x, \phi)$ is a finite set such that $is_{\Diamond}(x, \phi) \neq \emptyset$ iff $x \in var(\phi)$. Moreover, $\models \phi \leftrightarrow \bigwedge\{\Diamond^i x :\ x \in AF \ \&\ i \in is_{\Diamond}(x, \phi)\} \wedge \Diamond^{deg(\phi)}\top$.*

As before, if a finite set of pairs of formulas possesses a unifier then it possesses a closed unifier. Unlike the $\{\Box, \top, \wedge\}$-fragment, there exists non-unifiable finite sets of pairs of formulas. The truth is that many atom-free formulas are not equivalent to $\top$. Nevertheless, by Lemma 8.14, for all atom-free formulas ϕ, ϕ is equivalent to $\Diamond^{deg(\phi)}\top$.

Lemma 8.15 *Let ϕ be a formula. For all closed substitutions σ, $\models \sigma(\phi) \leftrightarrow \bigwedge\{\Diamond^i\sigma(x) :\ x \in var(\phi)\ \&\ i = \max is_\Diamond(x,\phi)\} \wedge \Diamond^{deg(\phi)}\top$.*

Lemma 8.16 *Let S be a finite set of pairs of formulas. Let ϕ, ψ, ϕ', ψ' be formulas such that $deg(\phi) = deg(\phi')$, $var(\phi) = var(\phi')$, $deg(\psi) = deg(\psi')$ and $var(\psi) = var(\psi')$. If for all $x \in AF$, $\max is_\Diamond(x,\phi) = \max is_\Diamond(x,\phi')$ and $\max is_\Diamond(x,\psi) = \max is_\Diamond(x,\psi')$ then $S \cup \{(\phi,\psi)\}$ possesses a unifier iff $S \cup \{(\phi',\psi')\}$ possesses a unifier.*

Let the normal formulas be defined as follows:

- $\phi ::= (x_1 \wedge \ldots \wedge x_\alpha) \mid \top \mid ((x_1 \wedge \ldots \wedge x_\alpha) \wedge \Diamond\phi) \mid \Diamond\phi$.

For example, the formula $\Diamond x \wedge \Diamond y$ is not normal and the formula $y \wedge \Diamond\Diamond z$ is normal. In the above definition of normal formulas, we use the conjunction $(x_1 \wedge \ldots \wedge x_\alpha)$ of the variables $x_1, \ldots, x_\alpha$. In such a situation, we will always consider that $\alpha \geq 1$. We shall say that a formula ϕ is minimalist if for all $x \in AF$, x occurs at most once in ϕ. For instance, the formula $\bigwedge\{\Diamond^i x :\ x \in var(\phi)\ \&\ i = \max is_\Diamond(x,\phi)\}$ is minimalist for each formula ϕ.

Lemma 8.17 *Let ϕ be a formula. There exists a normal formula ϕ' such that $\models \phi \leftrightarrow \phi'$. Moreover, if ϕ is minimalist then ϕ' is minimalist too. Finally, ϕ' can be easily computed from ϕ in polynomial time.*

For example, the non-normal formula $\Diamond x \wedge \Diamond y$ is equivalent to the normal formula $\Diamond(x \wedge y)$ and the non-normal formula $y \wedge \Diamond\top \wedge \Diamond\Diamond z$ is equivalent to the normal formula $y \wedge \Diamond\Diamond z$.

Lemma 8.18 *Let S be a finite set of pairs of formulas. There exists a finite set S' of pairs of minimalist normal formulas such that S possesses a unifier iff S' possesses a unifier. Moreover, S' can be easily computed from S in polynomial time.*

Let the thin formulas be defined as follows:

- $\phi ::= x \mid \top \mid (x \wedge \Diamond\phi) \mid \Diamond\phi$.

For example, the formula $\Diamond x \wedge \Diamond y$ is not thin and the formula $y \wedge \Diamond\Diamond z$ is thin. Remark that for all formulas ϕ, if ϕ is thin then ϕ is normal.

Proposition 8.19 *Let S be a finite set of pairs of minimalist normal formulas with variables $x_1, \ldots, x_\alpha$. Let $\preceq$ be a total order on $1, \ldots, \alpha$. Let S' be a finite set of pairs of thin minimalist formulas obtained from S and $\preceq$ by replacing each conjunct of the form $(x_{\beta_1} \wedge \ldots \wedge x_{\beta_n})$ in S by x_β where $\beta = \max_\preceq\{\beta_1, \ldots, \beta_n\}$. Suppose S' possesses a closed unifier σ such that*

- *for all $\beta = 1 \ldots \alpha$, there exists $k_\beta \in \mathbb{N}$ such that $\sigma(x_\beta) = \Diamond^{k_\beta}\top$,*
- *for all $\beta, \gamma = 1 \ldots \alpha$, if $\beta \preceq \gamma$ then $k_\beta \leq k_\gamma$.*

Then σ is also a unifier of S.

Proof. Let $(x_{\beta_1} \wedge \ldots \wedge x_{\beta_n})$ be a conjunct in S and $\beta = \max_\preceq\{\beta_1, \ldots, \beta_n\}$. Let $k_{\beta_1}, \ldots, k_{\beta_n} \in \mathbb{N}$ be such that $\sigma(x_{\beta_1}) = \Diamond^{k_{\beta_1}}\top$, …, $\sigma(x_{\beta_n}) = \Diamond^{k_{\beta_n}}\top$. Since for all $\gamma, \delta = 1 \ldots \alpha$, if $\gamma \preceq \delta$ then $k_\gamma \leq k_\delta$ and $\beta = \max_\preceq\{\beta_1, \ldots, \beta_n\}$, therefore

$k_\beta = \max_\preceq\{k_{\beta_1}, \ldots, k_{\beta_n}\}$. Hence, $\models \Diamond^{k_{\beta_1}}\top \wedge \ldots \wedge \Diamond^{k_{\beta_n}}\top \leftrightarrow \Diamond^{k_\beta}\top$. Thus, $\models \sigma(x_{\beta_1} \wedge \ldots \wedge x_{\beta_n}) \leftrightarrow \sigma(x_\beta)$. Since σ is a unifier of S', therefore σ is a unifier of S. $\square$

Proposition 8.20 *Let S be a finite set of pairs of minimalist normal formulas with variables $x_1, \ldots, x_\alpha$. Suppose S possesses a closed unifier σ such that*

- *for all $\beta = 1 \ldots \alpha$, there exists $k_\beta \in \mathbb{N}$ such that $\sigma(x_\beta) = \Diamond^{k_\beta}\top$.*

Let $\preceq$ be a total order on $1, \ldots, \alpha$ such that

- *for all $\beta, \gamma = 1 \ldots \alpha$, if $\beta \preceq \gamma$ then $k_\beta \leq k_\gamma$.*

Let S' be a finite set of pairs of thin minimalist formulas obtained from S and $\preceq$ by replacing each conjunct of the form $(x_{\beta_1} \wedge \ldots \wedge x_{\beta_n})$ in S by x_β where $\beta = \max_\preceq\{\beta_1, \ldots, \beta_n\}$. Then σ is also a unifier of S'.

Proof. Similar to the proof of Proposition 8.19. $\square$

In Propositions 8.19 and 8.20, the finite set S' of pairs of thin minimalist formulas obtained from S and $\preceq$ is called a thin $\preceq$-subset of S. Using the above results, a given finite set S of pairs of minimalist normal formulas with variables $x_1, \ldots, x_\alpha$ is unifiable iff there exists a total order $\preceq$ on $1, \ldots, \alpha$ and a thin $\preceq$-subset of S possessing a unifier. Now, in order to determine whether a given finite set S of pairs of thin minimalist normal formulas is unifiable, it suffices to consider the following procedure:

procedure $UNISET(S)$
begin
recursively replace each pair of the form $(\Diamond\phi, \Diamond\psi)$ in S by (ϕ, ψ)
$bool := BC(S)$
if $bool \wedge var(S) \neq \emptyset$ then
  begin
  guess a subset $AF(S)$ of $var(S)$
  for all $x \in AF(S)$ do
      replace in S each occurrence of x by $\top$
  for all $x \in var(S) \setminus AF(S)$ do
    replace in S each occurrence of x in S by $\Diamond x$
  transform S into an equivalent finite set of pairs of thin minimalist normal
    formulas
  $UNISET(S)$
  end
if $\neg bool$ then reject
accept
end

The function $BC(\cdot)$ takes as input a finite set S of pairs of thin minimalist normal formulas and returns the Boolean value

- $BC(S)$ ="if neither S contains pairs of the form $(\Diamond\phi, \top)$, nor S contains pairs of the form $(\top, \Diamond\psi)$ then $\top$ else $\bot$".

It can be implemented as a deterministic Turing machine working in polynomial time. The procedure $UNISET(\cdot)$ takes as input a finite set of pairs of thin minimalist normal formulas and accepts it iff it is unifiable. It can be implemented as a nondeterministic Turing machine working in polynomial space. Hence, the unification problem is in $NPSPACE$. Since $NPSPACE = PSPACE$, therefore

Proposition 8.21 *The unification problem is in $PSPACE$.*

Still, we do not know whether the unification problem is $PSPACE$-hard.

9 Conclusion

Much remains to be done. For example, there is the related admissibility problem: given an inference rule $\frac{\psi_1(x_1,\ldots,x_n),\ldots,\psi_k(x_1,\ldots,x_n)}{\chi(x_1,\ldots,x_n)}$, determine whether for all formulas $\phi_1,\ldots,\phi_n$, if $\models \psi_1(\phi_1,\ldots,\phi_n)$, $\ldots$, $\models \psi_k(\phi_1,\ldots,\phi_n)$ then $\models \chi(\phi_1,\ldots,\phi_n)$. One may also consider the unification problem when the ordinary modal language is extended by a set AP of parameters (denoted p, q, etc). In this case, the unification problem is to determine, given a formula $\psi(p_1,\ldots,p_\alpha,x_1,\ldots,x_\beta)$, whether there exists formulas $\phi_1,\ldots,\phi_\beta$ such that $\models \psi(p_1,\ldots,p_\alpha,\phi_1,\ldots,\phi_\beta)$. For each $k \geq 2$, one may also consider the unification problem in Alt_k, the least normal logic containing the formula $\Diamond(x_1 \wedge \neg x_2 \wedge \ldots \wedge \neg x_{k-1} \wedge \neg x_k) \wedge \ldots \wedge \Diamond(\neg x_1 \wedge \neg x_2 \wedge \ldots \wedge \neg x_{k-1} \wedge x_k) \rightarrow \Box(x_1 \vee x_2 \vee \ldots \vee x_{k-1} \vee x_k)$. Its decidability is open. Finally, what becomes of these problems when the ordinary modal language is extended by the master modality, the universal modality or the difference modality?

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Annex

Proof of Proposition 3.1: (i)$\Rightarrow$(ii): It suffices to remark that for all $n \in \mathbb{N}$, every n-valuation can be considered as a model based on a deterministic frame. (ii)$\Rightarrow$(i): It suffices to remark that for all $n \in \mathbb{N}$, every generated submodel of a model based on a deterministic frame is n-bisimilar to a k-valuation for some $k \in \mathbb{N}$ such that $k \leq n$.

Proof of Lemma 5.2: By induction on $\psi(x)$.

Proof of Lemma 5.3: Let $\phi \in AFF$. The following conditions are equivalent: (1) $\models \Diamond^k\top \to \psi(\phi)$; (2) for all $n \in \mathbb{N}$, $\models_n \Diamond^k\top \to \psi(\phi)$; (3) for all $n \in \mathbb{N}$, if $\models_n \Diamond^k\top$ then $\models_n \psi(\phi)$; (4) for all $n \in \mathbb{N}$, if $k \leq n$ then $(V_k(\phi, n, 0), \ldots, V_k(\phi, n, k)) \models \psi(x)$. The reasons for these equivalences to hold are the following: the equivalence between (1) and (2) follows from the definition of $\models$, the equivalence between (2) and (3) follows from the fact that $\phi \in AFF$ and the equivalence between (3) and (4) follows from Lemma 5.2.

Proof of Lemma 5.5: By definitions of $\equiv_k$ and f_k and Lemma 5.3.

Proof of Lemma 5.8: By Lemma 5.5 and Proposition 5.7.

Proof of Lemma 7.2: Suppose $k \leq l$. Let υ be the substitution defined by $\upsilon(x) = x \wedge \Box^k\bot$. The reader may easily verify that $\models \upsilon(\sigma_l(x)) \leftrightarrow \sigma_k(x)$. Hence, $\sigma_l \preceq \sigma_k$.

Proof of Lemma 7.3: Suppose $k < l$ and $\sigma_k \preceq \sigma_l$. Let υ be a substitution such that $\sigma_k \circ \upsilon \simeq \sigma_l$. Hence, $\models \upsilon(\sigma_k(x)) \leftrightarrow \sigma_l(x)$. Thus, $\models \Box^{<l}x \wedge \Box^l\bot \rightarrow \Box^k\bot$. Consequently, $\models \Box^l\bot \rightarrow \Box^k\bot$. Hence, $l \leq k$: a contradiction.

Proof of Lemma 8.1: Let θ and μ be the substitutions defined by

- $\theta(x) = x$ if $x \in var(\sigma(x))$ and $\theta(x) = \top$ otherwise,
- $\mu(x) = x$ if $x \in var(\tau(x))$ and $\mu(x) = \top$ otherwise.

The reader may easily verify that for all variables x, $\models \theta(\upsilon(x)) \leftrightarrow \sigma(x)$ and $\models \mu(\upsilon(x)) \leftrightarrow \tau(x)$. Hence, $\upsilon \preceq \sigma$ and $\upsilon \preceq \tau$.

Proof of Lemma 8.2: By induction on ϕ.

Proof of Lemma 8.3: By induction on ϕ.

Proof of Lemma 8.4: By induction on ϕ.

Proof of Lemma 8.5: By induction on ϕ.

Proof of Lemma 8.8: By Lemmas 8.2 and 8.7.

Proof of Lemma 8.9: Suppose $k \leq l$. Let υ be the substitution defined by $\upsilon(t_i) = t_k$ if $i = k$ and $\upsilon(t_i) = \top$ otherwise. The reader may easily verify that $\models \upsilon(\gamma_l(x)) \leftrightarrow \sigma_k(x)$, $\models \upsilon(\gamma_l(y)) \leftrightarrow \sigma_k(y)$ and $\models \upsilon(\gamma_l(z)) \leftrightarrow \sigma_k(z)$. Hence, $\gamma_l \preceq \sigma_k$.

Proof of Lemma 8.10: Suppose $k \leq l$. Let υ be the substitution defined by $\upsilon(t_i) = t_i$ if $i \leq k$ and $\upsilon(t_i) = \top$ otherwise. The reader may easily verify that $\models \upsilon(\gamma_l(x)) \leftrightarrow \gamma_k(x)$, $\models \upsilon(\gamma_l(y)) \leftrightarrow \gamma_k(y)$ and $\models \upsilon(\gamma_l(z)) \leftrightarrow \gamma_k(z)$. Hence, $\gamma_l \preceq \gamma_k$.

Proof of Lemma 8.12: By Lemma 8.9 and Proposition 8.11.

Proof of Lemma 8.14: By induction on ϕ.

Proof of Lemma 8.15: The equivalence between $\sigma(\phi)$, $\bigwedge\{\Diamond^i\sigma(x)$:

$x \in AF$ & $i \in is_\diamond(x,\phi)\} \wedge \diamond^{deg(\phi)}\top$ and $\bigwedge\{\diamond^i\sigma(x) : x \in var(\phi)$ & $i \in is_\diamond(x,\phi)\} \wedge \diamond^{deg(\phi)}\top$ is a consequence of Lemma 8.14. The equivalence between $\bigwedge\{\diamond^i\sigma(x) : x \in var(\phi)$ & $i \in is_\diamond(x,\phi)\} \wedge \diamond^{deg(\phi)}\top$ and $\bigwedge\{\diamond^i\sigma(x) : x \in var(\phi)$ & $i = \max is_\diamond(x,\phi)\} \wedge \diamond^{deg(\phi)}\top$ is a consequence of the fact that for all $i,j \in \mathbb{N}$, if $i \leq j$ then $\models \diamond^i\top \wedge \diamond^j\top \leftrightarrow \diamond^j\top$.

Proof of Lemma 8.16: Suppose for all $x \in AF$, $\max is_\diamond(x,\phi) = \max is_\diamond(x,\phi')$ and $\max is_\diamond(x,\psi) = \max is_\diamond(x,\psi')$. Let σ be a closed substitution. By Lemma 8.15, $\models \sigma(\phi) \leftrightarrow \bigwedge\{\diamond^i\sigma(x) : x \in var(\phi)$ & $i = \max is_\diamond(x,\phi)\} \wedge \diamond^{deg(\phi)}\top$, $\models \sigma(\phi') \leftrightarrow \bigwedge\{\diamond^i\sigma(x) : x \in var(\phi')$ & $i = \max is_\diamond(x,\phi')\} \wedge \diamond^{deg(\phi')}\top$, $\models \sigma(\psi) \leftrightarrow \bigwedge\{\diamond^i\sigma(x) : x \in var(\psi)$ & $i = \max is_\diamond(x,\psi)\} \wedge \diamond^{deg(\psi)}\top$ and $\models \sigma(\psi') \leftrightarrow \bigwedge\{\diamond^i\sigma(x) : x \in var(\psi')$ & $i = \max is_\diamond(x,\psi')\} \wedge \diamond^{deg(\psi')}\top$. Since $deg(\phi) = deg(\phi')$, $var(\phi) = var(\phi')$, $deg(\psi) = deg(\psi')$, $var(\psi) = var(\psi')$ and for all $x \in AF$, $\max is_\diamond(x,\phi) = \max is_\diamond(x,\phi')$ and $\max is_\diamond(x,\psi) = \max is_\diamond(x,\psi')$, therefore $\models \sigma(\phi) \leftrightarrow \sigma(\phi')$ and $\models \sigma(\psi) \leftrightarrow \sigma(\psi')$. Hence, $S \cup \{(\phi,\psi)\}$ possesses a unifier iff $S \cup \{(\phi',\psi')\}$ possesses a unifier.

Proof of Lemma 8.18: Let S' be the finite set of pairs of minimalist formulas obtained by replacing each pair (ϕ,ψ) in S of formulas by the pair $(\bigwedge\{\diamond^i x : x \in var(\phi)$ & $i = \max is_\diamond(x,\phi)\} \wedge \diamond^{deg(\phi)}\top, \bigwedge\{\diamond^i x : x \in var(\psi)$ & $i = \max is_\diamond(x,\psi)\} \wedge \diamond^{deg(\psi)}\top)$ of minimalist formulas. By Lemma 8.16, S possesses a unifier iff S' possesses a unifier. By Lemma 8.17, S' can be easily transformed into an equivalent finite set of pairs of minimalist normal formulas.